Random Entropy and Recurrence

Karma Dajani and Ronald Meester

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Abstract

We show that a cocycle (which is nothing but a generalized random walk with index set $\mathbf{Z}^d$) is recurrent whenever its random entropy is zero, and transient whenever its random entropy is positive. This generalizes a well known one-dimensional result, and implies a Polya type dichotomy for this situation.

1 Motivation and introduction

In Burton, Dajani and Meester (1996), the concept of *random entropy* associated with a $\mathbf{Z}^d$ random group action was introduced and studied. Every such $\mathbf{Z}^d$ random group action is generated via a cocycle. (For the readers with a probabilistic background, a cocycle is a generalization of an ordinary random walk. The main difference is the fact that cocycles are generally indexed by $\mathbf{Z}^d$ rather than $\mathbf{Z}$. A one-dimensional cocycle is nothing but an ordinary random walk; we give precise definitions later.) In the one-dimensional case it is easy to see that having positive random entropy is equivalent to transience of the associated random walk. It therefore seems reasonable to try to connect the concept of random entropy as developed in Burton, Dajani and Meester (1996) and the transience of the generating cocycle. In this paper we show that the one-dimensional connection holds in general.

The paper is completely self-contained. The next section contains the set-up, including all necessary definitions, and the main results. The last two sections contain the proofs.

2 Cocycles and random entropy

For ease of notation and description we will stick with the two-dimensional case. Everything we say goes through in all dimensions.

Let Ω be the following set:

$$\Omega = \{\omega = ((\omega_z^1, \omega_z^2)_{z \in \mathbf{Z}^2});\ \omega_z^i \in \mathbf{Z}^2,\ \text{and}\ \omega_z^1 + \omega_{z+e_1}^2 = \omega_z^2 + \omega_{z+e_2}^1\},$$

where e_i denote the unit vectors in $\mathbf{Z}^2$. You should think of ω_z^1 as the label of the edge between z and $z+e_1$, and of ω_z^2 as the label of the edge between z and $z+e_2$. The set Ω should be interpreted as follows: for two vertices z and z', let π be a edge-self-avoiding path from z to z'. Travelling from z to z' along π, we add all labels of edges which we traverse upwards or to the right, and subtract the labels of the edges which we traverse downwards or to the left. The property in the definition of Ω asserts that the outcome $g(z, z', \omega)$ is independent of the choice of π, and only depends on z and z' (and on ω of course). We define $f(z, \omega)$ to be $g(0, z, \omega)$. Then f is a map from $\mathbf{Z}^2 \times \Omega \to \mathbf{Z}^2$ and if $\phi : \mathbf{Z}^2 \times \Omega \to \Omega$ is the group action given by the coordinate shift, then f satisfies the cocycle identity

$$f(z + z', \omega) = f(z, \omega) + f(z', \phi_z(\omega)).$$

The cocycle f plays the role of the position of the random walk in the one-dimensional case, and the labels of the edges play the role of the increments. We let μ be a ϕ-invariant ergodic probability measure on Ω (on the natural σ-algebra) with the property that

$$\int_\Omega \|(\omega_0^1, \omega_0^2)\|_\infty d\mu(\omega) < \infty,$$

where $\|\cdot\|_\infty$ denotes the L_∞ norm.

Let F be a finite set containing at least two elements, and consider a $\mathbf{Z}^2$ action ψ on $X = F^{\mathbf{Z}^2}$, together with a ψ-invariant, ergodic measure ρ on X. The cocycle f induces a $(\mu \times \rho)$-invariant $\mathbf{Z}^2$-action $\Phi : \mathbf{Z}^2 \times \Omega \times X \to \Omega \times X$ as follows:

$$\Phi_z(\omega, x) = (\phi_z(\omega), \psi_{f(z,\omega)}(x)).$$

We continue with the definition of *random entropy*. We write $h_m(\xi)$ for the usual ergodic theoretical entropy with measure m and $\mathbf{Z}^2$-action ξ.

Definition 2.1 *The random entropy $E_\rho(\mu)$ is defined by*

$$E_\rho(\mu) = h_{\mu\times\rho}(\Phi) - h_\mu(\phi).$$

This notion was defined and studied in Burton, Dajani and Meester (1996). We can interpret the definition as follows. If $H \subset \mathbf{Z}^2$ is a finite subset, let P_H be the partition on Ω specifying the coordinates of $\omega \in \Omega$ indexed by elements of H. Similarly, we let Q_H denote the partition on X specifying the coordinates of $x \in X$ that are indexed by elements in H. We also write $B_n = \{0, 1, \ldots, n-1\}^2$.

Let, for $M \geq 0$,

$$L^M(n)(\omega) = \{u \in \mathbf{Z}^2; u = u' + u'', u' \in \{-M, \ldots, M\}^2, u'' \in f(B_n, \omega)\},$$

and let $S^M(n)$ denote the cardinality of $L^M(n)$. We also define the partitions

$$A_M = (P_{\{0\}} \times X) \vee (\Omega \times Q_{\{-M,\ldots,M\}^2}).$$

With a slight abuse of notation we write P_H instead of $P_H \times X$ and Q_S instead of $\Omega \times Q_S$. It will be clear from the context which space is considered. Then, using the entropy addition formula from Ward and Zhang (1992) in the second equality below, we have

$$\begin{aligned} h_{\mu\times\rho}(\Phi) &= \lim_{M\to\infty} \lim_{n\to\infty} \frac{1}{n^2} H_{\mu\times\rho}(\vee_{g\in B_n} \Phi_g^{-1}(A_M)) \\ &= \lim_{M\to\infty} \lim_{n\to\infty} \left\{ \frac{1}{n^2} H_\mu(P_{B_n}) + \frac{1}{n^2} H_\rho(Q_{L^M(n)} | P_{B_n}) \right\} \\ &= h_\mu(\phi) + \lim_{M\to\infty} \lim_{n\to\infty} \frac{1}{n^2} H_\rho(Q_{L^M(n)} | P_{B_n}). \end{aligned}$$

It follows that the random entropy $E_\rho(\mu)$ satisfies

$$E_\rho(\mu) = \lim_{M\to\infty} \lim_{n\to\infty} \frac{1}{n^2} H_\rho(Q_{L^M(n)} | P_{B_n}).$$

In words, to compute the random entropy, one looks in the box B_n, moves the square $\{-M, \ldots, M\}^2$ around according to the values of the cocycle in the box, computes the entropy of the corresponding partition, and finally divides by n^2 and takes the limit for $n \to \infty$. The answer will be independent of ω, and the limit for $M \to \infty$ then corresponds to taking the supremum over all partitions in the classical definition of entropy.

As mentioned before, a one-dimensional cocycle is just an ordinary random walk. If this random walk is simple, with X_i equal to 1 with probability p and -1 with probability $1-p$, then the random entropy can be computed and turns out to be equal to $|2p-1|\log 2$ (see Kifer (1986) and Burton, Dajani and Meester (1996)).

Finally, we need to define the notions of recurrence and transience of a cocycle.

Definition 2.2 *The cocycle f (or the measure μ) is said to be* recurrent *if*

$$\mu(\omega;\ f(z,\omega)=0 \textit{ for infinitely many } z \in \mathbf{Z}^d) = 1.$$

The cocycle f (or the measure μ) is said to be transient *if for all $z' \in \mathbf{Z}^d$ we have*

$$\mu(\omega;\ f(z,\omega)=f(z',\omega) \textit{ for infinitely many } z \in \mathbf{Z}^d) = 0.$$

In words, recurrence means infinitely many 'visits' to the origin a.s., and transience means that each image vector is attained only finitely many times a.s. It does not follow from the definitions that any given cocycle is either recurrent or transient, though we shall now see that this is the case nevertheless.

Theorem 2.3 *Suppose that $0 < h_\rho(\psi) < \infty$.*

1. *If $E_\rho(\mu) > 0$, then μ is transient.*

2. If $E_\rho(\mu) = 0$, then μ is recurrent.

Corollary 2.4 (A Polya dichotomy) *Any measure μ on Ω is either recurrent or transient.*

Proof: Given a measure μ on Ω, take a measure ρ on X with finite positive entropy and apply Theorem 2.3. $\square$

3 Preliminary results

Before we start proving anything, we remark that we shall go back and forth between probabilistic language and ergodic-theoretical language, depending which is more suitable for the current purpose. So for instance, we will use the phrases 'one-dimensional cocycle' and 'random walk' interchangebly. Also, sometimes we behave like probabilists and don't write the dependence on ω, but occasionaly it is convenient to stress this dependence.

We define horizontal and vertical limits as follows, writing $f = (f_1, f_2)$:

$$h_1(k) = \lim_{n\to\infty} \frac{f_1(n,k) - f_1(0,k)}{n}, \quad h_2(k) = \lim_{n\to\infty} \frac{f_2(n,k) - f_2(0,k)}{n},$$

$$v_1(k) = \lim_{n\to\infty} \frac{f_1(k,n) - f_1(k,0)}{n}, \quad v_2(k) = \lim_{n\to\infty} \frac{f_2(k,n) - f_2(k,0)}{n}.$$

All these limits exist μ a.e. by stationarity. We first claim that $h_1(k)$ is independent of k and similarly for the other quantities. To see this, we write X_n for $f_1(n,k) - f_1(0,k)$ and Y_n for $f_1(n,k+1) - f_1(0,k+1)$. We have that

$E|X_n - Y_n| \leq K$ for some uniform $K > 0$. (This follows from the integrability condition on μ.) Hence,

$$E\left(\left|\frac{X_n}{n} - \frac{Y_n}{n}\right|\right) \to 0$$

for $n \to \infty$ and it follows from Markov's inequality that $|\frac{X_n}{n} - \frac{Y_n}{n}|$ converges to 0 in probability and hence the a.e. limit (which we know exists) has to be 0 as well. This proves the claim. It follows that $h_1(k)$ is invariant under both horizontal and vertical translations and hence it is μ a.e. constant. Similar statements are valid for the other quantities. Therefore it makes sense to define $h_1 = h_1(k)$, $h_2 = h_2(k)$, $v_1 = v_1(k)$ and $v_2 = v_2(k)$. We write $u = (h_1, h_2)$ and $v = (v_1, v_2)$. The following result is taken from Burton, Dajani and Meester (1996) and identifies the random entropy:

Theorem 3.1 *We have*

$$E_\rho(\mu) = |\det(u, v)| h_\rho(\psi).$$

Next we state and prove some preliminary results needed for the proofs in the next section. The first result shows that we have convergence in measure for the values of the cocycle in any given direction.

Lemma 3.2 *Let $\{(k_n, m_n)\}$ be a sequence of vectors in $\mathbf{Z}^2$.*

(i) Suppose that $(k_n, m_n) \to (c_1 \cdot \infty, c_2 \cdot \infty)$ for some $c_1, c_2 \in \{1, -1\}$ and in addition that $\frac{m_n}{k_n} \to \alpha \in [-\infty, \infty]$. Then

$$\frac{f(k_n, m_n)}{|k_n| + |m_n|} \to \frac{c_1}{1 + |\alpha|} u + \frac{c_2 ||alpha|}{1 + |\alpha|} v$$

in μ measure as $n \to \infty$. (The quotient $\frac{1}{1+|\infty|}$ is to be interpreted as 0 and $\frac{\infty}{1+|\infty|}$ as 1.)

(ii) Suppose that $\{k_n\}$ is bounded and $m_n \to c_3 \cdot \infty$ for some $c_3 \in \{1, -1\}$. Then

$$\frac{f(k_n, m_n)}{|k_n| + |m_n|} \to c_3 v$$

in μ measure as $n \to \infty$.

(iii) Suppose that $\{m_n\}$ is bounded and $k_n \to c_4 \cdot \infty$ for some $c_4 \in \{1, -1\}$. Then

$$\frac{f(k_n, m_n)}{|k_n| + |m_n|} \to c_4 u$$

in μ measure as $n \to \infty$.

Proof: For (i), we will only prove the case $c_1 = c_2 = 1$ (and hence $\alpha \geq 0$) since the proofs of the other cases are all similar. Let $\epsilon > 0$, choose $\epsilon_1 > 0$ so that

$$\epsilon_1 \left(\frac{1}{1+\alpha} + \epsilon_1 \right) + |u|\epsilon_1 + \epsilon_1 \left(\frac{\alpha}{1+\alpha} + \epsilon_1 \right) + |v|\epsilon_1 < \epsilon.$$

(The reason for this complicated expression will become apparent soon.) Let

$$A(n, \epsilon_1) = \left\{ \omega; \left| \frac{f(n, 0, w)}{n} - u \right| < \epsilon_1 \right\}$$

and

$$B(n, \epsilon_1) = \left\{ \omega; \left| \frac{f(0, n, w)}{n} - v \right| < \epsilon_1 \right\}.$$

Using the convergence in measure (we have at this point in fact a.s. convergence), there exists N sufficiently large so that for all $n > N$,

(a) $\mu\left(A(k_n, \epsilon_1)\right) > 1 - \epsilon$,

(b) $\mu\left(B(m_n, \epsilon_1)\right) > 1 - \epsilon$,

(c)
$$\left|\frac{1}{1+\frac{m_n}{k_n}} - \frac{1}{1+\alpha}\right| < \epsilon_1,$$

and

(d)
$$\left|\frac{\frac{m_n}{k_n}}{1+\frac{m_n}{k_n}} - \frac{\alpha}{1+\alpha}\right| < \epsilon_1$$

hold. Since μ is translation invariant, for $n > N$ we have $\mu\left(\phi_{(n,0)}B(m_n, \epsilon_1)\right)$ $> 1-\epsilon$ and hence $\mu\left(A(k_n, \epsilon_1) \cap \phi_{(n,0)}B(m_n, \epsilon_1)\right) > 1 - 2\epsilon$. But for each $n > N$, we have that $A(k_n, \epsilon_1) \cap \phi_{(n,0)}B(m_n, \epsilon_1)$ is contained in the set

$$\left\{\omega; \left|\frac{f(k_n, m_n, \omega)}{k_n + m_n} - \frac{1}{1+\alpha}u - \frac{\alpha}{1+\alpha}v\right| < \epsilon\right\},$$

since for $\omega \in A(k_n, \epsilon_1) \cap \phi_{(n,0)}B(m_n, \epsilon_1)$ we have

$$\begin{aligned}
&\left|\frac{f(k_n, m_n, \omega)}{k_n + m_n} - \frac{1}{1+\alpha}u - \frac{\alpha}{1+\alpha}v\right| \\
\leq\ & \left|\frac{f(k_n, 0, \omega)}{k_n(1+\frac{m_n}{k_n})} - \frac{1}{1+\frac{m_n}{k_n}}u\right| + \left|\frac{1}{1+\frac{m_n}{k_n}} - \frac{1}{1+\alpha}\right| |u| \\
+\ & \left|\frac{f(0, m_n, \phi_{(k_n,0)}\omega)}{m_n(1+\frac{k_n}{m_n})} - \frac{1}{1+\frac{k_n}{m_n}}v\right| + \left|\frac{\frac{m_n}{k_n}}{1+\frac{m_n}{k_n}} - \frac{\alpha}{1+\alpha}\right| |v| \\
<\ & \epsilon_1(\frac{1}{1+\alpha} + \epsilon_1) + |u|\epsilon_1 + \epsilon_1(\frac{\alpha}{1+\alpha} + \epsilon_1) + |v|\epsilon_1 < \epsilon,
\end{aligned}$$

where the last inequality follows from the choice of ϵ_1. Thus, for all $n > N$

$$\mu\left(\{\omega; \left|\frac{f(k_n, m_n, \omega)}{k_n + m_n} - \frac{1}{1+\alpha}u - \frac{\alpha}{1+\alpha}v\right| < \epsilon\}\right) > 1 - 2\epsilon.$$

For (ii), recall that $f(0,n)/n$ converges in measure to v. It is clear that from this it follows that $f(-n,0)/n$ converges in measure to $-v$. Now we can write

$$\frac{f(k_n, m_n)}{|k_n| + |m_n|} = \frac{f(0, m_n)}{|m_n|} \cdot \frac{|m_n|}{|k_n| + |m_n|} + \frac{f(k_n, 0, \phi_{(0,m_n)}\omega)}{|k_n| + |m_n|}.$$

The first term converges to $c_3 v$ and the second term goes to zero in probability since $\{k_n\}$ is bounded, using the stationarity of μ.

The proof of (iii) is similar and is omitted. $\square$

Looking at the proof of the last lemma, we see that in fact we proved a somewhat stronger result. We have shown that we have uniform convergence in probability in a given direction. We formulate this for case (i) with $c_1 = c_2 = 1$ as follows:

Lemma 3.3 *Let $\alpha \in (-\infty, \infty)$. Then for any $\epsilon > 0$, there exist $N_\epsilon > 0$ and $\delta_\epsilon > 0$ such that whenever $m_n, k_n > N_\epsilon$ and $|m_n/k_n - \alpha| < \delta_\epsilon$, then*

$$\mu\left(\left|\frac{f(k_n, m_n)}{|k_n| + |m_n|} - \frac{c_1}{1+|\alpha|}u - \frac{c_2|\alpha|}{1+|\alpha|}v\right| > \epsilon\right) < \epsilon.$$

When $\alpha = \pm\infty$, δ_ϵ should be replaced by a constant M_ϵ and the condition $|m_n/k_n - \alpha| < \delta_\epsilon$ should be replaced by $|m_n/k_n| > M_\epsilon$.

Similar statements are valid for all other cases of Lemma 3.2.

For our next lemma, we need some additional notation. For each integer n define half planes as follows:

$$H_1(n) = \{(x, y) \in \mathbf{R}^2; y \leq n\},$$

$$H_2(n) = \{(x, y) \in \mathbf{R}^2; x \leq n\},$$

$$H_3(n) = \{(x, y) \in \mathbf{R}^2; y \geq n\},$$

and

$$H_4(n) = \{(x,y) \in \mathbf{R^2}; x \geq n\}.$$

Lemma 3.4 *Suppose $E_\rho(\mu) > 0$. Then there exist random variables N_1, N_2, N_3 and N_4, taking values in the positive integers, such that*

$$f(0,n) \notin f(H_1(0))\,\forall\, n \geq N_1,$$

$$f(n,0) \notin f(H_2(0))\,\forall\, n \geq N_2,$$

$$f(0,n) \notin f(H_3(0))\,\forall\, n \leq -N_3,$$

$$f(n,0) \notin f(H_4(0))\,\forall\, n \leq -N_4.$$

Proof: We will only prove the existence of N_1 since the other cases are proved similarly.

We call $\omega \in \Omega$ *very bad* if there exists a sequence $\{z_k\}$ in $H_1(0)$ and an infinite sequence $0 < n_1 < n_2 < \cdots$ of positive integers such that $f(z_k,\omega) = f(0,n_k,\omega)$ for all $k \geq 1$. Let

$$B = \{\omega\,;\, \omega \text{ is very bad }\}.$$

Our aim is to show that $\mu(B) = 0$. A problem here is that B is not clearly translation invariant. To overcome this difficulty we enlarge the set B as to get an invariant set.

We call ω *bad* if there exists an infinite set of distinct points $W = \{w_1, w_2, \ldots\} \in \mathbf{Z}^2 \backslash H_1(0)$ and a set Z=$\{z_1, z_2, \ldots\}$ of lattice points in $H_1(\ell)$, for *some* ℓ, such that

(i) $f(z_k,\omega) = f(w_k,\omega)$,

(ii) $\frac{w_{k,2}}{w_{k,1}} \to \infty$,

where $w_k = (w_{k,1}, w_{k,2})$. It is clear that the set

$$A = \{\omega\,;\,\omega \text{ is bad}\}$$

is translation invariant, and hence $\mu(A)$ is either 0 or 1. Furthermore, we have $B \subseteq A$, so that it suffices to prove that $\mu(A) = 0$. We now assume that $\mu(A) = 1$ and show that we get a contradiction.

The first thing to do is to select in a particular way a (random) subsequence of $(w_1, w_2, \ldots)$ which converges a.s. as follows. First order all points of $\mathbf{Z}^2$ in some deterministic way and order the points of W accordingly. According to Lemma 3.3 we can for all n find N_n and M_n such that for all w_k with $w_{k,2}/w_{k,1} > M_n$ and $|w_k| > N_n$ we have (writing $\|z\|$ for $|z_1| + |z_2|$)

$$\mu\left(\left|\frac{f(w_k)}{\|w_k\|} - v\right| > 2^{-n}\right) < 2^{-n}.$$

Now let, for all n, w_{k_n} be the first point in W which satisfies these conditions. Note that this choice is random in that $w_{k_n} = w_{k_n}(\omega)$. If we had to stress the dependence upon ω of the random variable

$$\frac{f(w_{k_n})}{\|w_{k_n}\|}$$

we would have to write

$$\frac{f(w_{k_n}(\omega), \omega)}{\|w_{k_n}(\omega)\|}.$$

It follows from the exponential rate of convergence that

$$\frac{f(w_{k_n})}{\|w_k\|} \to v$$

almost surely. For convenience, we relabel the indices and assume we have a (random) sequence $(w_1, w_2, \ldots)$ for which this almost sure convergence takes place. From now on, Z refers to the set of z_k's corresponding to the new labels, that is, we still have $f(w_k) = f(z_k)$ for all k.

The next thing is to rule out the possibility of the set Z being finite. This is not hard. From (i) it follows that we can write

$$\frac{f(w_k)}{\|w_k\|} = \frac{f(z_k)}{\|z_k\|} \cdot \frac{\|z_k\|}{\|w_k\|}. \tag{1}$$

The left hand side converges a.s. to v which is not the zero vector by assumption. On the event that Z is bounded, the right hand side converges to the zero vector. Therefore Z is unbounded a.s.

Next we let $a(z_k)$ be the angle that the vector z_k makes with the positive x-axis, measured counterclockwise. We define $\Theta = \Theta(\omega)$ as the (random) set of limit points of $\{a(z_k)\}$. Since the z_k's are all in $H_1(\ell)$ for some ℓ, we have that Θ is nonempty and satisfies $\Theta \subseteq [\pi, 2\pi]$. Since Θ is also closed, we can define

$$\bar{\theta} = \sup \Theta.$$

Note that Θ is clearly translation invariant, and therefore $\bar{\theta}$ is an almost sure constant. Using Lemma 3.3 as above, we can find a random subsequence $(z_{k_1}, z_{k_2}, \ldots)$ (where, as above $z_{k_n} = z_{k_n}(\omega)$) such that

$$\frac{z_{k_n,2}}{z_{k_n,1}} \to \tan th\bar{e}ta$$

and

$$\frac{f(z_{k_n})}{\|z_{k_n}\|} = \frac{f(z_{k_n}(\omega), \omega)}{\|z_{k_n}(\omega)\|}$$

converges a.s. to $\beta_1 v + \beta_2 u$ for appropriate β_1 and β_2. We claim that $\beta_1 \leq 0$. To see this, note that in Lemma 3.2, either case (i) with $c_2 = -1$, case (ii) with $c_3 = -1$ or case (iii) without condition on c_4 applies. In all these cases, the coefficient of v in the limit is at most 0.

Using equation (1) again, with k_n replacing k, we see that the left hand side still converges a.s. to v, and that the first term on the right hand side converges a.s. to a different vector, which is either linearly independent of v or a non-positive multiple of v. (Here we have used the fact that u and v are linearly independent and the fact that $\beta_1 \leq 0$.) The second term is for all n a (random) positive number, and hence we have arrived at a contradiction. □

4 Proofs of main results

Proof of Theorem 2.3: First suppose $E_\rho(\mu) = 0$. There are two possibilities: (i) either u or v is the zero vector, or (ii) u and v are linearly dependent. If u or v is zero, say u then it follows from Lemma 3.2 that f_1 defined by

$$f_1(n) = f(n, 0)$$

has the property that

$$\frac{f_1(n)}{n} \to 0$$

in μ measure as $n \to \infty$. Since f_1 is a random walk with stationary increments, it is well known (see Dekking (1982) or Schmidt (1984)) that this implies that f_1 is recurrent which in turn implies that f is recurrent.

Next we assume u and v are nonzero. Then $v = \gamma u$ for some $0 \neq \gamma \in \mathbf{R}$. If $\gamma = \frac{p}{q} \in \mathbf{Q}$, then f_2 defined by

$$f_2(n) = f(np, -nq)$$

is a stationary random walk satisfying

$$\frac{f_2(n)}{n} \to 0$$

in μ measure as $n \to \infty$. As in the previous case it follows that f_2 is recurrent and so f is recurrent as well.

Finally suppose γ is irrational, and let $\beta = -\frac{1}{\gamma}$. We generalise the proof of the above cases. That is, we want to pick lattice points close to the line $y = \beta x$ (which is the 'recurrence direction') in such a way that the cocycle f evaluated at these lattice points gives a recurrent one-dimensional random walk with stationary increments. This will be possible if we enlarge our probability space. The idea is to move the origin by a uniform distance $\delta \in [0,1]$ in the vertical direction, and on each vertical $x = n$ line we pick the lattice point closest to the intersection of $y = \beta x + \delta$ with the line $x = n$. The values of the cocycle f (which is now a function of δ and ω) evaluated at these points will now be shown to be a random walk with stationary increments. In this case, it is easier to adapt the cocycle language rather than the probabilistic language.

Consider the space $[0,1] \times \Omega$ with the product σ-algebra and product measure $P \times \mu$ where on $[0,1]$ we have the usual Borel σ-algebra with P Lebesgue measure. Define $U : [0,1] \times \Omega \to [0,1] \times \Omega$ by

$$U(\delta, \omega) = \left((\delta + \beta) \bmod 1, \phi_{(1, \lfloor \delta + \beta \rfloor)} \omega \right).$$

Then U clearly is $P \times \mu$ invariant and

$$U^n(\delta, \omega) = \left((\delta + n\beta) \bmod 1, \phi_{(n, \lfloor \delta + n\beta \rfloor)}\omega\right).$$

Define $g : [0,1] \times \Omega \to \mathbf{Z^2}$ by

$$g(\delta, \omega) = f(1, \phi_{(1, \lfloor \delta + \beta \rfloor)}\omega),$$

and $h : \mathbf{Z} \times [0,1] \times \Omega \to \mathbf{Z^2}$ by

$$h(n, \delta, \omega) = \sum_{i=0}^{n-1} g\left(U^i(\delta, \omega)\right) = f(n, \phi_{(n, \lfloor \delta + n\beta \rfloor)}\omega).$$

Then, h is a cocycle for the $\mathbf{Z}$-action generated by U. Since $\lim_{n\to\infty} \frac{\lfloor \delta + n\beta \rfloor}{n} = \beta$ and $v = \alpha\, u = -\frac{1}{\beta} v$ it follows from Lemma 3.2 that for each $\delta \in [0,1]$,

$$\frac{f(n, \lfloor \delta + n\beta \rfloor)}{n + |\lfloor \delta + n\beta \rfloor|} \to \frac{1}{1+|\beta|} u + \frac{\beta}{1+|\beta|} v = 0$$

in μ measure as $n \to \infty$. Since $\frac{n}{n + |\lfloor \delta + n\beta \rfloor|} \to \frac{1}{1+|\beta|}$ this also implies that

$$\frac{f(n, \lfloor \delta + n\beta \rfloor)}{n} \to 0 \tag{2}$$

in μ measure as $n \to \infty$.

We claim that $\frac{h(n,\cdot,\cdot)}{n} \to 0$ in $P \times \mu$ measure as $n \to \infty$. To see this, let $\epsilon > 0$. According to (2), for each $\delta \in [0,1]$ there exists N_δ such that for all $n \geq N_\delta$,

$$\mu\left(\{\omega; \left|\frac{f(n, \lfloor \delta + n\beta \rfloor, \omega)}{n}\right| < \epsilon\}\right) > \sqrt{1-\epsilon}.$$

Also, there exists a constant M such that

$$P\left(\{\delta; N_\delta \leq M\}\right) > \sqrt{1-\epsilon}.$$

Let

$$C(n,\delta)=\{\omega;\left|\frac{f(n,\lfloor\delta+n\beta\rfloor,\omega)}{n}\right|<\epsilon\}$$

and

$$D=\{\delta;N_\delta\le M\}.$$

For $n\ge M$ we have

$$\begin{aligned}
&P\times\mu\left(\{(\delta,\omega);\left|\frac{h(n,\delta,\omega)}{n}\right|<\epsilon\}\right)\\
&\quad= P\times\mu\left(\{(\delta,\omega);\left|\frac{f(n,\lfloor\delta+n\beta\rfloor,\omega)}{n}\right|<\epsilon\}\right)\\
&\quad= P\times\mu\left(\{(\delta,\omega);\omega\in C(n,\delta)\}\right)\\
&\quad= \int_0^1\mu\left(C(n,\delta)\right)\,dP(\delta)\\
&\quad\ge \int_D\mu\left(C(n,\delta)\right)\,dP(\delta)\ge 1-\epsilon.
\end{aligned}$$

This proves the claim. Since h is a cocycle for the $\mathbf{Z}$-action generated by U, it follows as before that h is recurrent i.e.

$$P\times\mu\left(\{(\delta,\omega);h(n,\delta,\omega)=0\text{ for infinitely many }n\in\mathbf{Z}\}\right)=1.$$

Projecting on the second coordinate yields that for almost all δ we have

$$\mu\left(\{\omega;f(n,\lfloor\delta+n\beta\rfloor,\omega)=0\text{ for infinitely many }n\in\mathbf{Z}\}\right)=1.$$

In fact we need only one δ with this property; anyway, it follows that f is recurrent.

Next we need to show that μ is transient when $E_\rho(\mu)>0$. For this we define the following stochastic processes:

$$Y_n^1(k)\;=\;\min\{N\ge 0\,;\,f(k,-n+\ell)\notin f(H_1(-n)),\,\forall\ell\ge N\},$$

$$\begin{aligned}
Y_n^2(k) &= \min\{N \geq 0\,;\, f(-n+\ell, k) \notin f(H_2(-n)),\, \forall \ell \geq N\}, \\
Y_n^3(k) &= \min\{N \geq 0\,;\, f(k, n-\ell) \notin f(H_3(n)),\, \forall \ell \geq N\}, \\
Y_n^4(k) &= \min\{N \geq 0\,;\, f(n-\ell, k) \notin f(H_4(n)),\, \forall \ell \geq N\}.
\end{aligned}$$

The idea behind these definitions is the following: $Y_n^1(k)$ for instance,is a random variable that indicates how far we need to go into the box $[-n, n]^2$ from below in order to make sure that no value in the lower half plane $H_1(n)$ is seen on the vertical line $x = k$ further up.

It follows from Lemma 3.4 that $Y_n^i(k)$ is well defined and finite a.s. Specialising to Y_n^1, note that the distribution of $Y_n^1(0)$ is independent of n, and therefore $((Y_n^1(0))_n$ is a stationary process. Hence there a.s. exists an n_1 such that $Y_{n_1}^1(0) < n_1$. It follows from the constructions that this implies that for all $n \geq n_1$ we have $Y_n^1(0) < n$. For the other processes Y^2, Y^3 and Y^4 we find numbers n_2, n_3 and n_4 such that for all $n \geq n_i$ we have $Y_n^i(0) < n$, $i = 2, 3, 4$.

Next define the (random) set $A_n \subset [-n, n]^2$ as all points (z_1, z_2) in $[-n, n]^2$ with the property that

$$\begin{aligned}
-n + Y_n^2(z_1) &\leq z_1 \leq n - Y_n^4(z_1), \\
-n + Y_n^1(z_2) &\leq z_2 \leq n - Y_n^3(z_2).
\end{aligned}$$

For all $n > \max\{n_1, n_2, n_3, n_4\}$ we have that the origin is contained in the set A_n. This implies that for these values of n, the value $f(0, \omega) = 0$ of the cocycle taken at the origin is not taken at any point outside B_n.

It is not hard to adapt this argument to other vertices z' as well, and this implies that the cocycle is transient. $\square$

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Department of Mathematics, University of Utrecht, P.O. Box 80.010, 3508 TA Utrecht, The Netherlands.

dajani@math.ruu.nl, meester@math.ruu.nl