Functions with Non-isolated Singularities on Singular Spaces

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FUNCTIONS WITH NON-ISOLATED SINGULARITIES ON SINGULAR SPACES

Functies met Niet-Geïsoleerde Singulariteiten op Singuliere Ruimten

(met een samenvatting in het Netherlands)

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR AAN DE UNIVERSITEIT UTRECHT, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF. DR. H. O. VOORMA. INGEVOLGE HET BESLUIT VAN HET COLLEGE VAN DECANEN, IN HET OPENBAAR TE VERDEDINGEN OP MAANDAG 16 FEBRUARI 1998, DES MIDDAGS TE 16.15 UUR

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Introduction

We study germs of analytic functions defined on an analytic space X. We mainly pay attention to spaces with isolated singularities, and functions with one dimensional singular loci on the spaces. This is a generalization of the work by Siersma[S3], Pellikaan[P1 and 2], and Bruce and Roberts[BR]. For other references and a short summary of related results see the beginning of each section.

In §1.1 we first recall some basic facts about the subgroups of $\mathrm{Aut}(\mathbb{C}^m, 0)$ keeping some analytic spaces fixed, and the logarithmic vector fields along the spaces. Then we discuss shortly the definition of functions with non-isolated singularities on a space X. All the function germs on X with singular loci containing a fixed subspace Σ of X are described by the so called primitive ideal. In §1.2 we prove two theorems on the degree of determinacy which will be used in §1.4 and §1.5. Among the non-isolated singularities, important and easily treated are the ones with transversal A_1 type singularity along the branches of their singular loci. In §1.3 we prove some properties of transversal A_1 singularities and a formula for the tangent space of the orbit.

It turns out that some properties of functions with singular loci containing Σ on X depend not only on Σ and X, but also on how Σ lies in X, that is, the relative position of Σ and X. The number $\lambda(\Sigma X)$ does give us some information about this, but not always. As an example we, in §1.4, consider the case that Σ is a smooth curve and X a surface with simple singularity. By choosing Σ as the x-axis of the local coordinate system, we obtain the normal forms of the simple surface singularities with respect to Σ. When Σ varies on X, the normal forms of X relative to Σ are different, and also $\lambda(\Sigma X)$ changes. This gives a picture about how smooth curves are embedded in a surface with a simple singularity. In doing this we find there is no smooth curve on E_8 type surface singularity. We also give a geometric explanation of the number $\lambda(\Sigma X)$.

In §1.5, we classify mainly function germs with singular loci a smooth curve (called line singularities) on surfaces with simple singularities. When $\lambda(\Sigma X)$ is minimal, we classify the simple germs on surfaces in $\mathbb{C}^3$ with singularity type A_k. We also consider the classifications of germs with singular loci a smooth curve on (higher dimensional) hypersurfaces with simple singularities. For a Σ on an A_k surface X_k ($k > 4$) with $\lambda(\Sigma X_k)$ not maximal and not minimal, there are no stable germs. We give the normal forms of the functions that have minimal codimensions. This will be helpful later for determining the homotopy type of the Milnor fibre of functions with line singularities on A_k surfaces.

We study the deformations of non-isolated singularities on singular spaces in chapter 2. Practically, to find the generators of the primitive ideal is quite involved, and if X is not a weighted homogeneous *icis,* it is very difficult, since it is already not easy to find the generator set of D_X. In §2.1 we prove that in some cases, one can easily find the generator set of the primitive ideal. This is related to the study of the conormal module M of the defining ideal of Σ in X. Since X is not smooth, the interesting conormal modules are not free and not torsion free. We find that some properties of the torsion submodule $T(M)$ (its length is called the torsion number) and the torsion free factor module $N := M/T(M)$ of M are important for us, and M, $T(M)$, N, and the primitive ideal $\int \mathfrak{g}$ are closely related. Under some conditions, a formula for the torsion number is given.

In §2.2, we study the relationship of the Jacobian number, the relative Tjurina number, the torsion number and δ_f. The residual critical locus and its image, the residual discriminant, are also studied in this section. When N is a free $\mathcal{O}_\Sigma$-module, we find useful deformations of a function with singular locus a one dimensional *icis* on a singular space with *icis.* The continuity of the Jacobian number under deformations is still an open problem.

The topology of the Milnor fibre F of a function f with non-isolated singularities on X is studied in chapter 3. In §3.1, we first prove results on homology and homotopy bouquet decompositions of the fibre F_s of a good (or an excellent) deformation f_s of f by using the additivity of the vanishing homology and excisions. This helps us to reduce our study of F_s (which is homotopy equivalent to F) to that of a "smaller" part F^0 of F_s. Then we decompose this F^0 into a union of the so called central type F^c and a number of cells attached to F^c along its transversal slice. With this we can express the Euler characteristic of F by that of F^c and the number of D_∞ and A_1 points. Hence as soon as we know the topology of F^c, we know the topology of the Milnor fibre F of the original function. As applications of the theory, we recover several known results and determine the homotopy type of F when f defines a line singularity on a (hyper)surface X, where X is of $A_1, A_2, A_3, A_4, A_5, D_k (k \geq 4)$, or E_7 type singularity. And when $\lambda(\Sigma X)$ is minimal or maximal, we are able to determine the homotopy type of F when X is a (hyper)surface with A_k type singularity for all $k \geq 1$.

In order to have an efficient formula for the Euler characteristic of F in some special cases, we prove a Pellikaan type formula in §3.2 for the Euler characteristic of F when f defines a line singularity on a weighted homogeneous *icis* and the transversal singularity of f along the line is A_1. By using *Singular,* a computer program for singularity theory [GPS], this formula is really efficient.

In §3.3 , we consider the blowing ups of certain singularities, and characterize simple line singularities on the A_1 surface by their liftings under the resolution of the surface. We use the resolutions, and sometimes the partial resolutions to determine the 0-connectedness of the Milnor fibre and to compute the Euler characteristic and zeta functions of the Milnor fibres of the singularities.

As an application, we prove that the Milnor fibre F of a function f on a surface X with simple isolated singularity is a bouquet of circles when f defines a line singularity on X and the transversal singularity type of f along Σ is A_1. The number of the circles in F can be calculated by the Euler characteristic $\chi(F)$, which can be calculated by using either the formula in §3.2 or the partial resolution.

Chapter 1

Non-isolated Singularities on Singular Spaces

1.1 Preliminaries and the Primitive Ideal

Siersma [S3] has studied function germs on $\mathbb{C}^m$ with a fixed line in their singular loci. This has been generalized by Pellikaan in [P1 and 2]. The *primitive ideal* $\int \mathfrak{g}$ of the defining ideal $\mathfrak{g}$ of a reduced analytic space $\Sigma \subset \mathbb{C}^m$ is exactly the set of function germs having Σ in their singular loci. In this section, we define also the primitive ideal consisting of all function germs on a singular space X having a subspace Σ of X in their singular loci, this is a generalization of [P1 and 2]. We start with a short review of some preliminaries about some subgroups of $\mathcal{R} := \mathrm{Aut}(\mathbb{C}^m, 0)$, the group of all the local automorphisms of $(\mathbb{C}^m, 0)$, as this also serves to fix some notations and conventions which will be used in the sequel. We refer the reader to [P1] for details.

1.1.1 Let $\mathcal{O}_{\mathbb{C}^m}$ be the structure sheaf of $\mathbb{C}^m$. The stalk $\mathcal{O}_{\mathbb{C}^m,0}$ of $\mathcal{O}_{\mathbb{C}^m}$ at 0 is a local ring, consisting of germs at 0 of analytic functions on $\mathbb{C}^m$. The ring $\mathcal{O}_{\mathbb{C}^m,0}$ is often denoted by $\mathcal{O}_m$, $\mathcal{O}_0$ or simply by $\mathcal{O}$ when no confusion can be caused. The unique maximal ideal of $\mathcal{O}_m$ is denoted by $\mathfrak{m}_m$ or $\mathfrak{m}$. Let $\mathcal{O}(q)$ be the local ring of analytic function germs on $(\mathbb{C}^m, 0)$ with q parameters: $\mathcal{O}(q) = \{F : (\mathbb{C}^m \times \mathbb{C}^q, 0) \longrightarrow \mathbb{C}\}$. So $\mathcal{O}(q)$ is isomorphic to $\mathcal{O}_{m+q}$. Let $\mathfrak{m}(q)$ be the maximal ideal of $\mathcal{O}(q)$. Sometimes we identify $\mathcal{O}$ with its image in $\mathcal{O}(q)$ by the projection germ $(\mathbb{C}^m \times \mathbb{C}^q, 0) \longrightarrow (\mathbb{C}^m, 0)$.

Let $(X, 0) \subset (\mathbb{C}^m, 0)$ be the germ of a reduced analytic subspace X of $\mathbb{C}^m$ at 0 defined by an ideal $\mathfrak{h} = (h_1, \ldots, h_p)$ of $\mathcal{O}$, generated by $h_1, \ldots, h_p \in \mathcal{O}$. Let $\mathfrak{g}$ be the ideal generated by $g_1, \ldots, g_n \in \mathcal{O}$. The germ of the analytic space defined by $\mathfrak{g}$ at 0 is denoted by $(\Sigma, 0)$. Write $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}$ and $\mathcal{O}_\Sigma := \mathcal{O}/\mathfrak{g}$.

1.1.2 Let $\mathcal{R} = \mathrm{Aut}(\mathbb{C}^m, 0)$, and $\mathcal{R}^e$ the pseudo-group of analytic isomorphism germs $\varphi : (\mathbb{C}^m, 0) \longrightarrow \mathbb{C}^m$. Let $\mathcal{R}^e(q) = \{H : (\mathbb{C}^m \times \mathbb{C}^q, 0) \longrightarrow \mathbb{C}^m \mid h_t := H(-,t) \in \mathcal{R}^e$ for all $t \in \mathbb{C}^q$ in a neighborhood of $0\}$, $\mathcal{R}(q) = \{H \in \mathcal{R}^e(q) \mid h_t := H(-,t) \in \mathcal{R}\}$. For any ideal $\mathfrak{i} \subset \mathcal{O}$, let $\mathcal{R}_{\mathfrak{i}} = \{\varphi \in \mathcal{R} \mid \varphi^* \mathfrak{i} = \mathfrak{i}\}$. One can also define $\mathcal{R}_{\mathfrak{i}}(q), \mathcal{R}^e_{\mathfrak{i}}$ and $\mathcal{R}^e_{\mathfrak{i}}(q)$ as in [P1]§2.

Let $\mathcal{R}_{\Sigma X} := \mathcal{R}_{\mathfrak{h}} \cap \mathcal{R}_{\mathfrak{g}}$, consisting of all $\varphi \in \mathcal{R}$ preserving both $(X, 0)$ and $(\Sigma, 0)$. $\mathcal{R}^e_{\Sigma X} = \mathcal{R}^e_{\mathfrak{h}} \cap \mathcal{R}^e_{\mathfrak{g}}, \mathcal{R}_{\Sigma X}(q) = \mathcal{R}_{\mathfrak{h}}(q) \cap \mathcal{R}_{\mathfrak{g}}(q), \mathcal{R}^e_{\Sigma X}(q) = \mathcal{R}^e_{\mathfrak{h}}(q) \cap \mathcal{R}^e_{\mathfrak{g}}(q)$.

1.1.3 Let Der denote the $\mathcal{O}$-module of germs of analytic vector fields on $\mathbb{C}^m$ at 0. Then Der is a free $\mathcal{O}$-module with $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_m}$ as basis, where $z_1, \dots, z_m$ are the local coordinates of $(\mathbb{C}^m, 0)$. Der is a Lie algebra with the bracket defined by $[\xi, \eta] := \xi\eta - \eta\xi$ for all $\xi, \eta \in \mathrm{Der}$.

Let $\mathrm{Der}(q) := \mathrm{Der}_{\mathcal{O}_{\mathbb{C}^q}}$ be the $\mathcal{O}_{\mathbb{C}^q}$ derivations of $\mathcal{O}(q)$. $\mathrm{Der}(q)$ is a free $\mathcal{O}(q)$-module of rank m.

For a germ of analytic space $(X, 0)$ defined by the ideal $\mathfrak{h}$, $\mathrm{Der}_{\mathfrak{h}} = \{\xi \in \mathrm{Der} \mid \xi(\mathfrak{h}) \subset \mathfrak{h}\}$ is the $\mathcal{O}$-module of *logarithmic vector fields* along $(X, 0)$ which is a Lie subalgebra of Der (see [BR]). When $\mathfrak{h}$ is a radical ideal defining the analytic space X, $\mathrm{Der}_{\mathfrak{h}}$ is often denoted by D_X. Geometrically, D_X consists of all germs of vector fields that are tangent to the smooth part of X. When X is a weighted homogeneous *icis* (isolated complete intersection singularity), one can write down precisely all the generators of D_X (see [W]).

Example Let $z_1, \dots, z_m$ be the coordinates of $\mathbb{C}^m$ with weights $w_1, \dots, w_m$ respectively. If X is defined by a weighted homogeneous polynomial $h = 0$ in $\mathbb{C}^m$ and has an isolated singularity at 0, then D_X is generated over $\mathcal{O}$ by:

$$\begin{aligned} \xi_E &= w_1 z_1 \frac{\partial}{\partial z_1} + \cdots + w_m z_m \frac{\partial}{\partial z_1}, \quad \text{the Euler derivation,} \\ \xi_{ij} &= \frac{\partial h}{\partial z_j}\frac{\partial}{\partial z_i} - \frac{\partial h}{\partial z_i}\frac{\partial}{\partial z_j} \quad (1 \le i < j \le m), \text{ the trivial derivations, and} \\ \theta_k &= h\frac{\partial}{\partial z_k}, \quad k = 1, \dots, m. \qquad \Box \end{aligned}$$

By [P1]§2, we know that

$$T\mathcal{R}^e_X(q) = \left\{ \xi \in \mathrm{Der}(q) \mid \xi = \sum_{j=1}^m \xi_j \frac{\partial}{\partial z_j}, \xi_j \in \mathcal{O}(q), \xi(\mathfrak{h}) = \mathfrak{h} \right\},$$

$$T\mathcal{R}_X(q) = \left\{ \xi \in \mathrm{Der}(q) \mid \xi = \sum_{j=1}^{m} \xi_j \frac{\partial}{\partial z_j}, \xi_j \in \mathfrak{m}\mathcal{O}(q), \xi(\mathfrak{h}) = \mathfrak{h} \right\},$$

$$T\mathcal{R}^e_{\Sigma X}(q) = \left\{ \xi \in \mathrm{Der}(q) \mid \xi = \sum_{j=1}^{m} \xi_j \frac{\partial}{\partial z_j}, \xi_j \in \mathcal{O}(q), \xi(\mathfrak{h}) = \mathfrak{h}, \xi(\mathfrak{g}) = \mathfrak{g} \right\},$$

$$T\mathcal{R}_{\Sigma X}(q) = \left\{ \xi \in \mathrm{Der}(q) \mid \xi = \sum_{j=1}^{m} \xi_j \frac{\partial}{\partial z_j}, \xi_j \in \mathfrak{m}\mathcal{O}(q), \xi(\mathfrak{h}) = \mathfrak{h}, \xi(\mathfrak{g}) = \mathfrak{g} \right\}.$$

Especially $T\mathcal{R}^e_X = \mathrm{Der}_{\mathfrak{h}} = D_X$, and

$$T\mathcal{R}_X = \left\{ \xi \in \mathrm{Der} \mid \xi = \sum_{j=1}^{m} \xi_j \frac{\partial}{\partial z_j}, \xi_j \in \mathfrak{m}, \xi(\mathfrak{h}) = \mathfrak{h} \right\}.$$

Set

$$\Theta = T\mathcal{R}_{\Sigma X} = \{\xi \in \mathrm{Der} \mid \xi(\mathfrak{h}) \subset \mathfrak{h}, \xi(\mathfrak{g}) \subset \mathfrak{g}, \xi(\mathfrak{m}) \subset \mathfrak{m}\}.$$

For $f \in \mathcal{O}$ we have ideals:

$$J_X(f) = \{\xi(f) \mid \xi \in T\mathcal{R}^e_X\} = \{\xi(f) \mid \xi \in D_X\}, \quad J_{\Sigma X}(f) = \{\xi(f) \mid \xi \in \Theta\},$$

$$J^0_X(f) = \{\xi(f) \mid \xi \in T\mathcal{R}_X\} \quad \text{and} \quad J^e_{\Sigma X}(f) = \{\xi(f) \mid \xi \in T\mathcal{R}^e_{\Sigma X}\}.$$

We call $J_X(f)$ the *Jacobian ideal* of f. Obviously, when X is the whole space $\mathbb{C}^m$ and $\Sigma = \{0\}$, namely, $\mathfrak{h} = \{0\}$, and $\mathfrak{g} = \mathfrak{m}$, then $J_X(f) = J(f)$, the Jacobian ideal of f, and $J^0_X(f) = J_{\Sigma X}(f) = \mathfrak{m}J(f)$.

When $X = \mathbb{C}^{n+1}$ and Σ is the x-axis of $\mathbb{C}^{n+1}$, namely, $\mathfrak{g} = (y_1, \dots, y_n)$, then $J_X(f) = J(f)$ and $J_{\Sigma X}(f) = \tau_{\mathfrak{g}}(f) := \mathfrak{m}\left(\frac{\partial f}{\partial x}\right) + \mathfrak{g}\left(\frac{\partial f}{\partial y}\right)$ (see [S3]).

It is obvious that, for any $\xi \in \Theta$, the flow ϕ_t generated by ξ preserves both $(X, 0)$ and $(\Sigma, 0)$. Namely $\phi_t \in \mathcal{R}_{\Sigma X}$ for all t.

The reader can learn more properties about $\mathcal{R}_{\mathfrak{h}}$ and $\mathrm{Der}_{\mathfrak{h}}$ from [P1] and [P2].

1.1.4 Let $\mathcal{S} = \{S_\alpha\}$ be an analytic stratification of X, $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ an analytic function germ. If X is smooth, we write Σ_f for the critical locus of f. In this case, $\Sigma_f \subset \mathcal{V}(f) := f^{-1}(0)$(see [Lê 1] or [Mi]).

Definition(see [GM] or [Ma] 1.2) The *critical locus* $\Sigma^{\mathcal{S}}_f$ of f relative to the stratification $\mathcal{S}$ is the union of the critical loci of f restricted to each of the strata S_α, namely, $\Sigma^{\mathcal{S}}_f = \bigcup_\alpha \Sigma_{f|S_\alpha}$.

For an analytic space X embedded in a neighborhood U of $0 \in \mathbb{C}^m$, there is a logarithmic stratification which we roughly describe in the following (see

[BR] for details): By shrinking U if it is necessary, D_X determines a coherent subsheaf, denoted by $\mathcal{D}_X$, of the sheaf $\mathcal{D}\mathfrak{er}_U$ as an $\mathcal{O}_U$-module. There are a finite number of global sections ξ^j of $\mathcal{D}_X$ generating this $\mathcal{O}_U$-module. Since $\mathcal{D}_X$ is closed under Lie bracket, they yield an involutive system of vector fields. By Frobenius theorem, U can be decomposed into maximal integral submanifolds of U which form a stratification $\mathcal{S}_{\log} := \{X_\alpha\}$, *the logarithmic stratification* of U. The following properties of this stratification are of importance:

- For any $z \in X_\alpha$, $T_z X_\alpha = \{\xi(z) \mid \xi \in \mathcal{D}_{X,z}\}$, where $\xi(z)$ denotes the value of ξ at z;
- If X is pure dimensional, the components of $U \setminus X$, and $X_{\text{reg}} := X \setminus X_{\text{sing}}$ are logarithmic strata;
- In general, $\mathcal{S}_{\log}$ is not locally finite. If $\mathcal{S}_{\log}$ is locally finite, then X is said to be *holonomic.*

Let X be of pure dimension. The collection $\{X \cap X_\alpha \mid X_\alpha \in \mathcal{S}_{\log}\}$ is a stratification of X which will be called the *logarithmic stratification* of X in this article. Especially, when X is purely dimensional and has isolated singularity in 0, then $\{0\}$ and the connected components of $X \setminus \{0\}$ form a holonomic logarithmic stratification of X. So 0 is always a critical point of any germ $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ relative to this stratification.

Hence for $f \in \mathfrak{m}$, $\Sigma_f = \{p \in X \mid \xi(f)(p) = 0 \text{ for all } \xi \in D_X\}$ is the critical locus of f relative to the logarithmic stratification. Obviously $\Sigma_f = X \cap \mathcal{V}(J_X(f))$ (see [Mi] 2.7). If the dimension of Σ_f is not positive, we say that f defines (or has) *isolated singularities* on X. If the dimension of Σ_f is positive, we say that f defines (or has) *non-isolated singularities* on X. If Σ_f is one dimensional smooth complex manifold, we say that f defines (or has) a *line singularity* on X.

Remark that for a morphism $f : X \longrightarrow \mathbb{C}$ with X a space of pure dimension $n+1$, there is an intrinsic way to define Σ_f as the zero set of the n^{th} Fitting ideal, denoted by C_f, of the module of relative Kähler one-forms Ω^1_f. In fact C_f is generated by $n \times n$ minors of a presentation matrix of Ω^1_f. We call C_f the *critical ideal* of f (see [Lo]).

1.1.5 Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be an analytic function germ on an anlytic space germ $(X, 0)$ which is locally embedded in $\mathbb{C}^m$. It is known that for a generic linear form l, $f + l$ has only complex Morse singularities on X if f has isolated singularities (see [GM], [BR], [Lê 4]). And for $k >> 0$, $f_k := f + l^{k+1}$ (the series of f) has $\Sigma_{f_k} = \Sigma_f \cap l^{-1}(0)$ if f has non-isolated singularities on X

(see for example [Lê 5], or [Ti 2]). The statements above indicate a question: For what kind of linear form l, $f+l$ (or even l) defines "bad" singularities on X?

Example 1 Let $X : h = xy + \frac{1}{2}z^2 = 0$, $l = ax + by + cz$. Then Σ_l is defined by $h = ax - by = az - cy = bz - cx = 0$. We know that $\Sigma_l = \{0\}$ for generic a, b, c. Indeed $\dim\Sigma_l > 0$ if and only if $(a : b : c)$ satisfies $2ab + c^2 = 0$. This equation defines a curve B in $\mathbf{P}^2(\mathbb{C})$. In this case Σ_l is a line on X.

We consider the following question:

Given radical ideals $\mathfrak{h} \subset \mathfrak{g}$, defining X and Σ respectively. What is the set of functions on $(X, 0)$ having $(\Sigma, 0)$ in their singular loci?

Obviously, all functions from $\mathfrak{g}^2$ restricted to $(X, 0)$ have $(\Sigma, 0)$ in their singular loci. However the following example 2 shows that $\mathfrak{g}^2$ is too small for answering our question. Namely, there exists function $f \in \mathfrak{g} \setminus \mathfrak{g}^2$ such that $\Sigma \subset \Sigma_f$. But this function is not generic in $\mathfrak{g}$. By this we mean that if we disturb it in $\mathfrak{g}$ a little bit, the disturbed function will have isolated singularities on X(see [Lê1,2]).

Example 2 Take $h = xy + z^2, \mathfrak{g} = (y, z), f = y$, then f has the $x-$axis as its singular locus Σ_f. But $f_s = f + sy + tz$ has an isolated singularity in 0 when s, t take generic values.

1.1.6 Definition Let $\mathfrak{h} = (h_1, \ldots, h_p) \subset \mathfrak{g} = (g_1, \ldots, g_n)$ be radical ideals of $\mathcal{O}_{\mathbb{C}^m,0}$, $X = \mathcal{V}(\mathfrak{h}), \Sigma = \mathcal{V}(\mathfrak{g})$. Define a subset of $\mathcal{O}$, called the *primitive ideal* of $\mathfrak{g}$:

$$\int_{\mathfrak{h}} \mathfrak{g} := \{f \in \mathcal{O} \mid (f) + J_X(f) \subset \mathfrak{g}\}.$$

Example For $\mathfrak{g} = (y, z) \subset \mathcal{O}_{\mathbb{C}^3,0}$, if $\mathfrak{h}_1 = (z)$, then $\int_{\mathfrak{h}_1} \mathfrak{g} = (y^2, z)$, if $\mathfrak{h}_2 = (z^2)$, then $\int_{\mathfrak{h}_2} \mathfrak{g} = \mathfrak{g}$, and if $\mathfrak{h}_3 = (xy + z^2)$, then $\int_{\mathfrak{h}_3} \mathfrak{g} = (y, z^2)$. The images of these primitive ideals under the canonical projection of $\pi : \mathcal{O}_{\mathbb{C}^3,0} \longrightarrow \mathcal{O}_X = \frac{\mathcal{O}}{\mathfrak{h}}$ are $(\bar{y}^2), (\bar{y}, \bar{z})$ and $(\bar{y})$ respectively.

1.1.7 Remarks 1) When X is smooth this is the definition of Pellikaan (see [P1 and 2]). It is straightaway to verify that $\int_{\mathfrak{h}} \mathfrak{g}$ is an ideal of $\mathcal{O}$ and one always has $\mathfrak{g}^2 \subset \int_{\mathfrak{h}} \mathfrak{g} \subset \mathfrak{g}$ and $\mathfrak{h} \subset \int_{\mathfrak{h}} \mathfrak{g}$. And for $\mathfrak{g}_i \supset \mathfrak{h}$ $(i = 1, 2)$, we have $\int_{\mathfrak{h}} \mathfrak{g}_1 \cap \int_{\mathfrak{h}} \mathfrak{g}_2 = \int_{\mathfrak{h}} (\mathfrak{g}_1 \cap \mathfrak{g}_2)$;

2) The ideal $\int_{\mathfrak{h}} \mathfrak{g}$ depends on the ideal $\mathfrak{h}$ as well as $\mathfrak{g}$ (see the example in §1.1.6). In the following we always assume that $\mathfrak{h}$ and $\mathfrak{g}$ are radical and $\int_{\mathfrak{h}} \mathfrak{g}$

is denoted by $\int_X \mathfrak{g}$ or $\int \mathfrak{g}$ when no confusion can be caused by this;

3) The singular locus of f is $\Sigma_f := \mathcal{V}(J_X(f)) \cap X$, and $\Sigma_f \subset f^{-1}(0)$ if $f(0) = 0$. If $f \in \int \mathfrak{g}$, then $\Sigma \subset X \cap \mathcal{V}(J_X(f)) = \Sigma_f$. Conversely, for $f \in \mathfrak{m}$, we have $f \in \int \mathfrak{g}$ when $\Sigma \subset \Sigma_f$ and $\mathfrak{g}$ is radical. The reason is: $J_X(f) \subset \mathfrak{g}$, and since f takes finite values on Σ_f and $0 \in \Sigma$, $f|_\Sigma = 0$, so $f^k \in \mathfrak{g}$ for some $k \in \mathbb{N}$. Hence $f \in \mathfrak{g}$ since $\mathfrak{g}$ is radical;

4) By the Second Exact Sequence (see [H]II 8.12 or [Ma 26.I]), we have the exact sequence of $\mathcal{O}_\Sigma$-modules:

$$\frac{\mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}} \xrightarrow{\delta} \Omega^1_X \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0 \tag{1.1.7.1}$$

where $\delta(\bar{b}) = db \otimes 1$ and $d : \mathcal{O}_X \longrightarrow \Omega^1_X$ is the differential of $\mathcal{O}_X$.

When X is smooth, there is also an exact sequence (see [P1 and 2]):

$$0 \longrightarrow \frac{\mathfrak{g}}{\int \mathfrak{g}} \xrightarrow{\delta} \Omega^1_X \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0 \tag{1.1.7.2}$$

This is not the case if X is singular.

Example (This example is due to Pellikaan) Take $X : h = xy + z^2 = 0$. And Σ is defined by $\mathfrak{g} = (y, z)$. Then there is a presentation:

$$\mathcal{O}_X \xrightarrow{dh} \mathcal{O}^3_X \longrightarrow \Omega^1_X \longrightarrow 0,$$

where $dh = \begin{pmatrix} y & x & 2z \end{pmatrix}^T$ as a matrix. Tensoring with $\mathcal{O}_\Sigma$, we have exact sequence

$$\mathcal{O}_\Sigma \xrightarrow{\bar{dh}} \mathcal{O}^3_\Sigma \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma \longrightarrow 0$$

where we have $\bar{dh} = \begin{pmatrix} 0 & x & 0 \end{pmatrix}^T$ as a matrix. Hence $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma \cong \mathcal{O}^2_\Sigma \oplus \frac{\mathcal{O}_\Sigma}{(x)}$. However $\Omega^1_\Sigma \cong \mathcal{O}_\Sigma$ and $\frac{\mathfrak{g}}{\int \mathfrak{g}} \cong \mathcal{O}_\Sigma$. Hence (1.1.7.2) is not exact in this case. Note that the morphism δ in (1.1.7.1) is injective (see §2.1.5 Corollary 2). $\square$

1.1.8 If $\phi \in \mathcal{R}_{\Sigma X}$, then $\phi^*\mathfrak{h} \subset \mathfrak{h}$, $\phi^*\mathfrak{g} \subset \mathfrak{g}$. For any $f \in \int_X \mathfrak{g}$, we have $\phi^*(f) \in \mathfrak{g}$, $J_X(\phi^*(f)) = J_X(f \circ \phi) = \phi^* J_X(f) \subset \mathfrak{g}$. Hence $\mathcal{R}_{\Sigma X}$ acts on $\int_X \mathfrak{g}$. Namely, $\mathcal{R}_{\Sigma X} \subset \mathcal{R}_{(\int_X \mathfrak{g})} \cap \mathcal{R}_X$. Similar results hold for the tangent spaces: $T\mathcal{R}_{\Sigma X} \subset T\mathcal{R}_{\int_X \mathfrak{g}} \cap T\mathcal{R}_X$. we do not know when the equalities hold.

1.1.9 By §1.1.8, for $f \in \int \mathfrak{g}$, we have

$$J^e_{\Sigma X}(f) \subset J_X(f) \cap \int_X \mathfrak{g} \qquad \text{and} \qquad J_{\Sigma X}(f) \subset J^0_X(f) \cap \int_X \mathfrak{g}.$$

Definition

- Call $J_{\Sigma X}(f)+\mathfrak{h}$ the *tangent space* of f;
- Define
$$c(f)=\dim_{\mathbb{C}}\frac{\int_X \mathfrak{g}}{J_{\Sigma X}(f)+\mathfrak{h}} \qquad \text{and} \qquad c^e(f)=\dim_{\mathbb{C}}\frac{\int_X \mathfrak{g}}{J^e_{\Sigma X}(f)+\mathfrak{h}},$$
called the *codimension* and *extended codimension* of f on X respectively;
- Call $\frac{\mathfrak{g}}{J_X(f)+\mathfrak{h}}$ the *Jacobian module* of f on X, and its dimension over $\mathbb{C}$ is called the *Jacobian number* of f on X and is denoted by
$$j(f):=\dim_{\mathbb{C}}\frac{\mathfrak{g}}{J_X(f)+\mathfrak{h}}.$$

We have $c(f)\geq c^e(f)$. By Artin-Rees Lemma (cf. [AM]), $c(f)<\infty$ if and only if $c^e(f)<\infty$.

Remark that for $f\in\int\mathfrak{g}$, $\mathfrak{h},\mathfrak{g},\int\mathfrak{g},J_{\Sigma X}(f),J^e_{\Sigma X}(f)$, and $J_X(f)$ are $\mathcal{R}_{\Sigma X}$-invariant, so are $c(f),c^e(f)$ and $j(f)$.

1.2 Finite Determinacy

The theory of finite determinacy and unfoldings for map germs has been systematically developed by Arnol'd (see e.g. [AGV]) and Mather[M1 and 2]. Damon has made this axiomatic in [D1 and 2]. So whenever one wants to study this theory with respect to some group, the only thing one needs to do is to verify if the group is a geometric one. We shall not do this with our group $\mathcal{R}_{\Sigma X}$ here. Since we need the precise order of determinacy, we, in this section, prove two theorems on the degree of determinacy which will be used in §1.4 and §1.5 for classification.

1.2.1 Definition Let $\mathfrak{h} \subset \mathfrak{g}$ and X, Σ be the same as in §1.1.

1) Two germs $f, g \in \int_X \mathfrak{g}$ are *$\mathcal{R}_{\Sigma X}$-equivalent* if there exsits a $\phi \in \mathcal{R}_{\Sigma X}$ such that $f \circ \phi = g$ on X.

2) A germ $f \in \int_X \mathfrak{g}$ is *$k-\mathcal{R}_{\Sigma X}-$determined on X in $\int_X \mathfrak{g}$* (or *k-determined* for short) if for any $g \in \int_X \mathfrak{g}$ with $f - g \in \mathfrak{m}^{k+1} \cap \int_X \mathfrak{g}$ on X, then f and g are $\mathcal{R}_{\Sigma X}$-equivalent on X;

3) A germ $f \in \int_X \mathfrak{g}$ is *finitely determined* if there exists a $k \in \mathbb{N}$ such that f is $k - \mathcal{R}_{\Sigma X}-$determined.

1.2.2 By Artin-Rees Lemma (see [AM]), there exists an integer r such that $\mathfrak{m}^{k+r} \cap \int_X \mathfrak{g} = \mathfrak{m}^k(\mathfrak{m}^r \cap \int_X \mathfrak{g})$ for all non-negative integer k. Let r be the minimal number such that the above equality is true and write $M_r = \mathfrak{m}^r \cap \int_X \mathfrak{g}$. The following theorem is a generalisation of [S3](1.5) and [P1] 6.5 (see also [D1]).

Theorem Let $f \in \int_X \mathfrak{g}$,

1) If
$$\mathfrak{m}^{k+1} M_r \subset \mathfrak{m} J_{\Sigma X}(f) + \mathfrak{m}^{k+2} M_r + \mathfrak{h},$$
then f is $k + r$-determined in $\int_X \mathfrak{g}$;

2) If f is k-determined in $\int_X \mathfrak{g}$, then
$$\mathfrak{m}^{k+1} \cap \int_X \mathfrak{g} \subset J_{\Sigma X}(f) + \mathfrak{h}.$$

Proof 1) Let $g \in \int \mathfrak{g}$ satisfy $f - g \in \mathfrak{m}^{k+r+1} \cap \int \mathfrak{g} = \mathfrak{m}^{k+1} M_r$ on X. Define
$$F(z,t) = f(z) + t(g(z) - f(z)), \quad f_t(z) := F(z,t) \in \mathcal{O}(1).$$
Embed $\mathcal{O}_{\mathbb{C}^m,0} \subset \mathcal{O}(1)$, $\mathfrak{m}_m \subset \mathfrak{m}(1)$. Let
$$J^*_{\Sigma X}(F) = \{\xi(F) \mid \xi \in T\mathcal{R}_{\Sigma X}(1)\}.$$

It is follows from §1.1 that $\xi(g-f) \in \mathfrak{m}_m^{k+1}M_r\mathcal{O}(1)+\mathfrak{h}\mathcal{O}(1)$ for all $\xi \in T\mathcal{R}_{\Sigma X}(1)$. Since

$$\xi(f)=\xi(F)-t\xi(g-f)$$

we have

$$J_{\Sigma X}(f)\mathcal{O}(1) \subset J^*_{\Sigma X}(F)+\mathfrak{m}_m^{k+1}M_r\mathcal{O}(1)+\mathfrak{h}\mathcal{O}(1)$$

and

$$\begin{aligned}\mathfrak{m}_m^{k+1}M_r\mathcal{O}(1) &\subset \mathfrak{m}_m J_{\Sigma X}(f)\mathcal{O}(1)+\mathfrak{m}_m^{k+2}M_r\mathcal{O}(1)+\mathfrak{h}\mathcal{O}(1)\\ &\subset \mathfrak{m}(1)J^*_{\Sigma X}(F)+\mathfrak{m}(1)\mathfrak{m}_m^{k+1}M_r+\mathfrak{h}\mathcal{O}(1).\end{aligned}$$

By Nakayama's Lemma,

$$\mathfrak{m}_m^{k+1}\mathcal{O}(1)M_r \subset \mathfrak{m}(1)J^*_{\Sigma X}(F)+\mathfrak{h}\mathcal{O}(1).$$

Moreover,

$$\frac{\partial F}{\partial t}=g-f\in\int\mathfrak{g}\cap\mathfrak{m}_m^{k+r+1}=\mathfrak{m}_m^{k+1}M_r.$$

Hence $f-g=\xi(F)+G$ for some $\xi\in\Theta\mathcal{O}(1)$ and $G\in\mathfrak{h}$. Let $\Phi(x,y,t)$ be the flow of ξ. Define $\varphi_t(x,y)=\Phi(x,y,t)\in\mathcal{R}_{\Sigma X}(1)$, then $\varphi_t\in\mathcal{R}_{\Sigma X}$ (see [P1]). Thus $f=f_0=f_t\circ\varphi_t$ on X for all small t. By continuous induction over the interval $[0,1]$, we find that $g=f_1$ and $f=f_0$ are $\mathcal{R}_{\Sigma X}-$equivalent.

2) Note that $f+\mathfrak{m}^{k+1}\cap\int\mathfrak{g}+\mathfrak{h}\subset\mathcal{R}_{\Sigma X}(f)$, the $\mathcal{R}_{\Sigma X}$-orbit of f. Taking the tangent space at f and identifying $T_f\mathcal{O}$ with $\mathcal{O}$, we can get

$$\mathfrak{m}^{k+1}\cap\int\mathfrak{g}\subset T_f(\mathcal{R}_{\Sigma X})=J_{\Sigma X}(f)+\mathfrak{h}$$

since $\mathfrak{m}^{k+1}\cap\int\mathfrak{g}$ is a $\mathbb{C}$-vector space. □

1.2.3 Corollary $c(f)<\infty$ if and only if f is finitely determined. □

1.2.4 In §1.4 we will consider all the surfaces contain a fixed line Σ and an equivalence defined by all the coordinate transformations which preserve the fixed line. We recall some notations from [S3]. Let $\mathfrak{g}=(y_1,\dots,y_n)$ defines the x-axis Σ of $\mathbb{C}^{n+1}$. Consider the action of $\mathcal{R}_\Sigma$ on $\mathfrak{m}\mathfrak{g}$. As in §1.2.1, two germs $f,g\in\mathfrak{m}\mathfrak{g}$ are called $\mathcal{R}_\Sigma$-*equivalent* if there exists a $\phi\in\mathcal{R}_\Sigma$ such that $f=g\circ\phi$. A germ $f\in\mathfrak{m}\mathfrak{g}$ is called $k-\mathcal{R}_\Sigma$-*determined* in $\mathfrak{m}\mathfrak{g}$ if for each $g\in\mathfrak{m}\mathfrak{g}$ with $f-g\in\mathfrak{m}^{k+1}\cap\mathfrak{g}=\mathfrak{m}^k\mathfrak{g}$, f and g are $\mathcal{R}_\Sigma$-equivalent.

It is easy to prove the following theorem:

Theorem If

$$\mathfrak{m}^k\mathfrak{g} \subset \mathfrak{m}\tau_{\mathfrak{g}}(f) + \mathfrak{m}^{k+1}\mathfrak{g},$$

then f is $k - \mathcal{R}_\Sigma$-determined, where

$$\tau_{\mathfrak{g}}(f) = \mathfrak{m}\left(\frac{\partial f}{\partial x}\right) + \mathfrak{g}\left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}\right)$$

is the tangent space of the $\mathcal{R}_\Sigma$-orbit $\mathcal{R}_\Sigma(f)$ at f. $\square$

1.3 Transversal A_1 Singularities

For a function with non-isolated singularities, there are transversal singularities defined along branches of its singular locus. When X is smooth, Siersma [S3] and Pellikaan[P1, 2 and 4] have studied non-isolated singularities with transversal type A_1. T. de Jong [deJ] has studied line singularities with transversal type A_2, A_3, D_4, E_6, E_7, and E_8. In this section we prove that some results of Pellikaan and Siersma on transversal A_1 singularities remain true when the space X is singular.

1.3.1 Definition Let $(X, 0)$ be the germ of a reduced analytic space with isolated singularity in 0. Let Σ be a reduced curve germ on $(X, 0)$ defined by $\mathfrak{g}$ and have isolated singularity in 0. A germ $f \in \int \mathfrak{g}$ is called a *transversal A_1 singularity* along Σ on X if its singular locus $\Sigma_f = \Sigma$, and, for $P \in \Sigma \setminus 0, f$ has only A_1 singularity transversal to the branch of $\Sigma \setminus 0$ containing P(see [S3], [P1]).

1.3.2 Proposition For $f \in \int_X \mathfrak{g}$, the following statements are equivalent:
1) f is a transversal A_1 singularity along Σ on X;
2) $j(f) < \infty$;
3) $c(f) < \infty$;
4) f is finitely determined in $\int_X \mathfrak{g}$.

1.3.3 Example Before proving the proposition, we consider the smooth case which is essential for the proof of proposition 1.3.2. Let $x, y_1, \dots y_s, z_1, \dots, z_t$ be the coordinates of $\mathbb{C}^m$, $m = s + t + 1$. Let $\mathfrak{h} = (z_1, \dots, z_t)$ and $\mathfrak{g} = (y_1, \dots, y_s) + (z_1, \dots, z_t)$ define subspace $X = \mathbb{C}^{s+1} \subset \mathbb{C}^m$ and the x-axis Σ respectively. Then

$$\int_X \mathfrak{g} = (y_1, \dots, y_s)^2 + (z_1, \dots, z_t).$$

For $f \in \int_X \mathfrak{g}$, we have

$$\begin{aligned} J_X(f) &= \left\{\xi(f) \mid \xi \in T\mathcal{R}^e_{\mathfrak{h}}\right\} \\ &= \left\{\xi(f) \mid \xi = \xi_0 \frac{\partial}{\partial x} + \sum_{i=1}^{s} \eta_i \frac{\partial}{\partial y_i} + \sum_{j=1}^{t} \zeta_j \frac{\partial}{\partial z_j}, \zeta_j \in \mathfrak{h}\right\} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_s}\right) + \mathfrak{h}\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_t}\right), \end{aligned}$$

$$
\begin{aligned}
J^0_X(f) &= \{\xi(f) \mid \xi \in T\mathcal{R}_{\mathfrak{h}}\} \\
&= \left\{\xi(f) \mid \xi = \xi_0 \frac{\partial}{\partial x} + \sum_{i=1}^{s} \eta_i \frac{\partial}{\partial y_i} + \sum_{j=1}^{t} \zeta_j \frac{\partial}{\partial z_j}, \xi_0, \eta_i \in \mathfrak{m}, \zeta_j \in \mathfrak{h}\right\} \\
&= \mathfrak{m}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_s}\right) + \mathfrak{h}\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_t}\right).
\end{aligned}
$$

In the case of $t = 0$, we have $J_X(f) = J(f)$ and $J^0_X(f) = \mathfrak{m}J(f)$. We also have:

$$
\begin{aligned}
J^e_{\Sigma X}(f) &= \{\xi(f) \mid \xi \in T\mathcal{R}^e_{\Sigma X}\} \\
&= \left\{\xi(f) \mid \xi = \xi_0 \frac{\partial}{\partial x} + \sum_{i=1}^{s} \eta_i \frac{\partial}{\partial y_i} + \sum_{j=1}^{t} \zeta_j \frac{\partial}{\partial z_j}, \eta_i \in \mathfrak{g}, \zeta_j \in \mathfrak{h}\right\} \\
&= \left(\frac{\partial f}{\partial x}\right) + \mathfrak{g}\left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_s}\right) + \mathfrak{h}\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_t}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{\Sigma X}(f) &= \{\xi(f) \mid \xi \in T\mathcal{R}_{\Sigma X}\} \\
&= \left\{\xi(f) \mid \xi = \xi_0 \frac{\partial}{\partial x} + \sum_{i=1}^{s} \eta_i \frac{\partial}{\partial y_i} + \sum_{j=1}^{t} \zeta_j \frac{\partial}{\partial z_j}, \xi_0 \in \mathfrak{m}, \eta_i \in \mathfrak{g}, \zeta_j \in \mathfrak{h}\right\} \\
&= \mathfrak{m}\left(\frac{\partial f}{\partial x}\right) + \mathfrak{g}\left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_s}\right) + \mathfrak{h}\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_t}\right).
\end{aligned}
$$

From these we have:

$$
\begin{aligned}
\frac{\mathfrak{g}}{J_X(f) + \mathfrak{h}} &\cong \frac{(y_1, \dots, y_s)\mathcal{O}_{s+1}}{(\frac{\partial \bar{f}}{\partial x}, \frac{\partial \bar{f}}{\partial y_1}, \dots, \frac{\partial \bar{f}}{\partial y_n})\mathcal{O}_{s+1}}, \\
\frac{\int \mathfrak{g}}{J_{\Sigma X}(f) + \mathfrak{h}} &\cong \frac{(y_1, \dots, y_s)^2\mathcal{O}_{s+1}}{\mathfrak{m}_{s+1}(\frac{\partial \bar{f}}{\partial x}) + (y_1, \dots, y_s)(\frac{\partial \bar{f}}{\partial y_1}, \dots, \frac{\partial \bar{f}}{\partial y_n})\mathcal{O}_{s+1}},
\end{aligned}
$$

where $\bar{f}$ is the restricted function of f on $\mathbb{C}^{s+1}$. These coincide with the formulae for $\bar{f}$ in [S3].

1.3.4 Proof of 1.3.2 We take representatives for every germs involved. Define sheaves of $\mathcal{O}$-modules $\mathcal{F}^1$ and $\mathcal{F}^2$ as follows:

$$
\mathcal{F}^1(U) = \frac{\mathfrak{g}}{J_X(f) + \mathfrak{h}}, \qquad \mathcal{F}^2(U) = \frac{\int \mathfrak{g}}{J_{\Sigma X}(f) + \mathfrak{h}},
$$

where U is an open neighbourhood of 0 in $\mathbb{C}^m$, and $\mathfrak{g}, \int \mathfrak{g}, \mathfrak{h}, J_X$ and $J_{\Sigma X}$ are considered as modules over $\mathcal{O}(U)$.

By a version of the Nullstellensatz for coherent sheaves (see e.g. [Wa1](2.2)) and example 1.3.3, one can easily prove 2) $\Longleftrightarrow$ 1) $\Longleftrightarrow$ 3). The point is to investigate when the sheaves are supported in $\{0\}$. 3) $\Longleftrightarrow$ 4) is Corollary 1.2.3. $\square$

1.3.5 Proposition Let $\mathfrak{h}$ and $\mathfrak{g}$ be the same as in 1.3.1. If $f \in \int_X \mathfrak{g}$ is a transversal A_1 singularity along Σ on X, then

$$J^e_{\Sigma X}(f) = J_X(f) \cap \int_X \mathfrak{g}, \qquad J_{\Sigma X}(f) = J^0_X(f) \cap \int_X \mathfrak{g}, \qquad \text{and } c^e(f) \leq j(f).$$

Proof By 1.1.9,

$$J^e_{\Sigma X}(f) \subset J_X(f) \cap \int_X \mathfrak{g} \quad \text{and} \quad J_{\Sigma X}(f) \subset J^0_X(f) \cap \int_X \mathfrak{g}.$$

Take a neighborhood U of 0 in $\mathbb{C}^m$ such that $X \cap U$ is defined by analytic functions $h_1, \ldots, h_p$ in U, and $U \cap \Sigma$ is defined by analytic functions $g_1, \ldots, g_n$ in U. Let $\mathcal{H}$ be the ideal sheaf of $X \cap U$ generated on U by $h_1, \ldots, h_p$, $\mathcal{G}$ the ideal sheaf of $\Sigma \cap U$ generated on U by $g_1, \ldots, g_n$. Let U be so small that the representatives of the generators $\xi^1, \ldots \xi^s$ of $\mathcal{O}$-module D_X are well defined and generate a sheaf denoted by $\mathcal{D}_X$. There is a presentation of $\mathcal{D}_X$ as $\mathcal{O}$-modules

$$\mathcal{O}^s \xrightarrow{\pi} \mathcal{D}_X \longrightarrow 0 \tag{1.3.5.1}$$

and by tensoring with $\mathcal{O}_\Sigma$ we get

$$\mathcal{O}^s_\Sigma \xrightarrow{\pi} \mathcal{D}_X \otimes \mathcal{O}_\Sigma \longrightarrow 0 \tag{1.3.5.2}$$

as $\mathcal{O}_\Sigma$-modules, where π is defined by $\pi(\lambda_1, \ldots, \lambda_s) = \sum_{j=1}^{s} \lambda_j \xi^j$.

Let $\int \mathcal{G}$ be the sheaf defined by $\int \mathfrak{g}$ and $f \in \int \mathcal{G}$ represents $f \in \int \mathfrak{g}$. Let sheaves $\mathcal{J}^e_{\Sigma X}(f)$, $\mathcal{J}_{\Sigma X}(f)$, $\mathcal{J}_X(f)$ and $\mathcal{J}^0_X(f)$ be defined by ideals $J^e_{\Sigma X}(f)$, $J_{\Sigma X}(f)$, $J_X(f)$ and $J^0_X(f)$ respectively.

Since $f \in \int \mathcal{G}$, we have $\xi^j(f) = \sum_{k=1}^{n} \varphi_{jk} g_k$, with $\varphi_{jk} \in \mathcal{O}$, $j = 1, \ldots, s, k = 1, \ldots n$. We get matrices

$$\varphi = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{s1} \\ \cdots & \cdots & \cdots \\ \varphi_{1n} & \cdots & \varphi_{sn} \end{pmatrix}, \qquad D^2_f = \begin{pmatrix} \xi^1\xi^1(f) & \cdots & \xi^1\xi^s(f) \\ \cdots & \cdots & \cdots \\ \xi^s\xi^1(f) & \cdots & \xi^s\xi^s(f) \end{pmatrix},$$

$$D_g = \begin{pmatrix} \xi^1(g_1) & \cdots & \xi^1(g_n) \\ \cdots & \cdots & \cdots \\ \xi^s(g_1) & \cdots & \xi^s(g_n) \end{pmatrix}.$$

These matrices define mappings:

$$\varphi : \mathcal{O}_\Sigma^s \longrightarrow \mathcal{O}_\Sigma^n, \quad D_f^2 : \mathcal{O}_\Sigma^s \longrightarrow \mathcal{O}_\Sigma^s, \quad D_g : \mathcal{O}_\Sigma^n \longrightarrow \mathcal{O}_\Sigma^s.$$

D_f^2 and D_g induce mappings:

$$d_f^2 : \mathcal{D}_X \otimes \mathcal{O}_\Sigma \longrightarrow \mathcal{D}_X \otimes \mathcal{O}_\Sigma, \quad d_g : \mathcal{O}_\Sigma^n \longrightarrow \mathcal{D}_X \otimes \mathcal{O}_\Sigma.$$

Since for any $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathcal{O}_\Sigma^s$, we have

$$\xi^j \left(\sum_{i=1}^{s} \lambda_i \xi^i(f) \right) \equiv \sum_{i=1}^{s} \lambda_i \xi^j \xi^i(f) \quad (\bmod \mathfrak{g}).$$

$$\begin{tikzcd}
\mathcal{O}_\Sigma^s \arrow[rr, "D^2 f"] \arrow[dr, "\varphi"] \arrow[dd, "\pi"] & & \mathcal{O}_\Sigma^s \arrow[dd, "\pi"] \\
& \mathcal{O}_\Sigma^n \arrow[ur, "D_g"] \arrow[dr, "d_g"] & \\
\mathcal{D}_X \otimes \mathcal{O}_\Sigma \arrow[rr, "d^2 f"] \arrow[d] & & \mathcal{D}_X \otimes \mathcal{O}_\Sigma \arrow[d] \\
0 & & 0
\end{tikzcd}$$

Diagram 1.3.1

It follows that diagram 1.3.1 is commutative, where $\pi(\lambda) = \sum_{j=1}^{s} \lambda_j \xi^j$ for $\lambda = (\lambda_1, \dots, \lambda_s) \in \mathcal{O}_\Sigma^s$, and the vertical sequences are exact. By taking $\mathcal{H}\mathfrak{om}_{\mathcal{O}_\Sigma}$, we get commutative diagram 1.3.2.

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
\mathcal{H}om(\mathcal{D}_X \otimes \mathcal{O}_\Sigma, \mathcal{O}_\Sigma) \arrow[rr, "(d^2 f)^*"] \arrow[dd, "\pi^*"'] \arrow[dr, "(d_g)^*"] & & \mathcal{H}om(\mathcal{D}_X \otimes \mathcal{O}_\Sigma, \mathcal{O}_\Sigma) \arrow[dd, "\pi^*"] \\
& \mathcal{H}om(\mathcal{O}_\Sigma^n, \mathcal{O}_\Sigma) \arrow[dr, "\varphi^*"] & \\
\mathcal{H}om(\mathcal{O}_\Sigma^s, \mathcal{O}_\Sigma) \arrow[ur, "(D_g)^*"] \arrow[rr, "(D^2 f)^*"'] & & \mathcal{H}om(\mathcal{O}_\Sigma^s, \mathcal{O}_\Sigma)
\end{tikzcd}$$

Diagram 1.3.2

Now we mimic Pellikaan's arguments. Note that $(\mathcal{K}er(d_g)^*) \subset (\mathcal{K}er(d^2 f)^*)$. For $P \neq 0$, f has only transversal A_1 singularity at P, hence

$$(\mathcal{K}er(d^2 f)^*)_P = (\mathcal{K}er(d_g)^*)_P.$$

Let

$$\tilde{u} \in \mathcal{K}er(d^2 f)^* \subset \mathcal{H}om(\mathcal{D}_X \otimes \mathcal{O}_\Sigma^s, \mathcal{O}_\Sigma^s).$$

Outside the origin, we have $(d_g)^*(\tilde{u}) = 0$. Let $u = \pi^*(\tilde{u})$, then $(D_g)^*(u) = 0$. Namely the composition function:

$$\Sigma \quad \xrightarrow{u} \quad \mathbb{C}^s \quad \xrightarrow{(D_g)^*} \quad \mathbb{C}^n$$

is zero outside the origin. Hence $(D_g)^*(u) = 0$ everywhere. This proves

$$\mathcal{K}er(d_g)^* \supset \mathcal{K}er(d^2 f)^*.$$

Let $\psi = \zeta(f) \in J_X(f) \cap \int \mathfrak{g}$ for some $\zeta = \sum_{k=1}^{s} u_k \xi^k \in D_X$. Since $f \in \int \mathfrak{g}$, we have $\xi^k(f) \in \mathfrak{g}$. Since $\psi \in \int \mathfrak{g}$,

$$\xi^j(\psi) \equiv \sum_k^s u_k \xi^j \xi^k(f) \equiv 0 \quad (\bmod \mathfrak{g}).$$

So if we view u as an element in $\mathcal{H}\mathfrak{om}(\mathcal{D}_X \otimes \mathcal{O}_\Sigma, \mathcal{O}_\Sigma)$ such that

$$u\left(\sum_{k=1}^{s} \lambda_k \xi^k\right) = \sum_{k=1}^{s} \lambda_k u(\xi^k) = \sum_{k=1}^{s} \lambda_k u_k,$$

then $u \in \mathcal{K}\mathfrak{er}(d^2 f)^* = \mathcal{K}\mathfrak{er}(d_g)^*$. So for any $t = (t_1, \ldots, t_n) \in \mathcal{O}_\Sigma^n$, we have

$$\begin{aligned} 0 &\equiv (d_g)^*(u)(t) = u(\pi(D_g(t))) \\ &\equiv u\left[\sum_{k=1}^{s}\left(\sum_{j=1}^{n} t_j \xi^k(g_j)\right)\xi^k\right] \\ &\equiv \sum_{j=1}^{n} t_j \left(\sum_{k=1}^{s} u_k \xi^k(g_j)\right) \quad (\bmod \mathfrak{g}). \end{aligned}$$

Hence

$$\zeta(g_j) = \sum_{k=1}^{s} u_k \xi^k(g_j) \in \mathfrak{g} \quad (j = 1, \ldots n)$$

and

$$\zeta = \sum_{k=1}^{s} u_k \xi^k \in D_\Sigma \cap D_X.$$

This proves $\psi \in J^e_{\Sigma X}(f)$.

The other equality can be proved similarly.

Since $(J_X(f) + \mathfrak{h}) \cap \int \mathfrak{g} = J_X(f) \cap \int \mathfrak{g} + \mathfrak{h} = J^e_{\Sigma X}(f) + \mathfrak{h}$, we have

$$\frac{\mathfrak{g}}{J_X(f) + \mathfrak{h}} \supset \frac{J_X(f) + \mathfrak{h} + \int \mathfrak{g}}{J_X(f) + \mathfrak{h}} = \frac{\int \mathfrak{g}}{J^e_{\Sigma X}(f) + \mathfrak{h}}.$$

This proves the inequality. □

1.4 Lines on Surfaces with Simple Singularities

Simple (hyper)surface singularities have been known for long time since they occur in several different situations. They have at least fifteen characterizations (see [Du]). One often describes them by the normal forms of their defining equations in a certain coordinate system. This has been done by Siersma [S1 and 2] and Arnol'd (see *e.g.* [AGV]). The idea is to classify all the surfaces that contain the origin 0 by the right equivalence defined by the group $\mathcal{R}$, consisting of all the coordinate transformations preserving 0. Among the equivalence classes are those without parameters in their normal forms, called the simple surface singularities, or the $A-D-E$ singularities.

In this section, we take a different but similar point of view toward two dimensional hypersurface singularities. Take a line Σ as the x-axis of the coordinate system in $\mathbb{C}^3$, classify all the surfaces that contain Σ under the equivalence defined by the subgroup $\mathcal{R}_\Sigma$ of $\mathcal{R}$, consisting of all the coordinate transformations preserving Σ. Choosing different lines on a surface X (if there exist) as the x-axis, we look at the normal forms of X. This classifies the lines on the surface X. We give an $\mathcal{R}_\Sigma$ invariant λ to tell the differences between the lines. In our case, this λ is equal to the torsion number, which will be defined and studied in §2.1.

1.4.1 Lines on singular spaces Let Σ be a smooth curve in $\mathbb{C}^{n+1}$. Since locally Σ is isomorphic to a line, we often say that Σ is a line. Let X be a singular space embedded in $\mathbb{C}^{n+1}$. For a singular point $O \in X_{\text{sing}}$, if there exists a line (smooth curve) Σ in $\mathbb{C}^{n+1}$ such that $O \in \Sigma$ and $\Sigma \setminus \{O\} \subset X_{\text{reg}}$, we say that X *contains (or has) smooth curve* passing through O. Very often, we say that X *contains (or has) a line* passing through O.

Let $\mathcal{R}_\Sigma$ be the group of all the automorphisms of $\mathbb{C}^{n+1}$ that preserve Σ. Take Σ as the x-axis in $\mathbb{C}^{n+1}$, then the defining ideal of Σ is $\mathfrak{g} := (y_1, \dots, y_n)$, and

$$\mathcal{R}_\Sigma = \{\phi \in \mathcal{R} \mid \phi^* \mathfrak{g} \subset \mathfrak{g}\}.$$

It has an action on $\mathfrak{mg}$ from the right hand side. This defines an equivalence relation on $\mathfrak{mg}$. The usual machinery for subgroups of $\mathcal{R}$ applies:

a) There is a theorem on the determinacy degree with respect to $\mathcal{R}_\Sigma$ and $\mathfrak{mg}$ (cf. §1.2.4);

b) A natural concept of $\mathcal{R}_\Sigma$-codimension.

Moreover, for a germ $h \in \mathfrak{mg}$, there exists also the next natural $\mathcal{R}_\Sigma$-

invariant

$$\lambda := \lambda(h) = \dim_{\mathbb{C}} \frac{\mathcal{O}}{J(h)+\mathfrak{g}},$$

where $J(h)$ is the Jacobian ideal of h. Note that

$$1 \leq \lambda(h) \leq \mu(h),$$

where $\mu(h)$ is the Milnor number of h.

Remark We will return to this $\lambda = \lambda(h)$ in §2.1. In case Σ is a line on an *icis* X, we will show that λ is equal to the *torsion number* $\varrho(\Sigma X)$ of Σ and X, which is the length of the torsion submodule of the conormal module M of $\bar{\mathfrak{g}}$ in the ring $\mathcal{O}_X$. Moreover, λ only depends on Σ and X. Hence very often we denote λ by $\lambda(\Sigma X)$, and call it the torsion number.

1.4.2 Lines on singular surfaces On a singular surface X one can not always find a line passing through (not contained in) the singular locus of X. Gonzalez-Sprinberg and Lejeune-Jalabert [GL1,2], by using resolutions of singularities, have proved a criterion for the existence of lines on X (see §3.3.1 for a quotation of this criterion). Especially, on surfaces with A_k, $D_k (k \geq 4)$, E_6 or E_7 type singularity, there exist lines passing through the singular point. In fact one can find easily lines on this kind of surfaces simply by looking at the defining equations. But it is not easy to tell how many different lines there are on a surface, and what kind of invariants can be used to tell the differences. The hardest thing is to prove the non-existence of smooth curve on E_8 type surface (*loc.cit.*).

1.4.3 Simple surface singularities Let X be a surface germ with isolated singularity at the origin. The singularity of X is *simple* if and only if X can be defined in a neighborhood of the origin in $\mathbb{C}^3$ by the following equations(see *e.g.* [S2]):

$$\begin{array}{ll} A_k: & x^{k+1}+y^2+z^2=0 \quad (k \geq 0) \\ D_k: & x^2y+y^{k-1}+z^2=0 \quad (k\geq 4) \\ E_6: & x^4+y^3+z^2=0 \\ E_7: & x^3y+y^3+z^2=0 \\ E_8: & x^5+y^3+z^2=0 \end{array}$$

The left hand sides of the equations are called the normal forms of the simple (or A-D-E) singularities.

1.4.4 Classification of lines on surfaces with simple singularities

We are going to prove the main result of this section.

Theorem A surface $X \subset \mathbb{C}^3$ with a simple isolated singularity contains a line passing through its singular point if and only if it is of type:

$$A_k \quad (k \geq 1), \quad D_k \quad (k \geq 4), \quad E_6, \quad \text{or} \quad E_7.$$

Moreover, if we choose the singular point of X as the origin and the line Σ on X as the x-axis in $\mathbb{C}^3$, that is, Σ is defined by the ideal $\mathfrak{g} = (y, z)$, then the defining equation of X is $\mathcal{R}_\Sigma$-equivalent to one of the equations in table 1.

Table 1 : Simple Singularities Passing Through x-axis

Type of X	Equations	λ	Name
A_k $(k\geq 1)$	$xy + z^{k+1} = 0$	1	$A_{k,1}$
	or		
	$x^l y + x^{s+1}z^2 + yz = 0 \quad (k = 2l + s, l \geq 2, s \geq 0)$,	l	$A_{k,l}$
	or		
	$x^l y + y^2 + z^2 = 0 \quad (k = 2l - 1, l \geq 2)$	l	$A_{k,l}$
D_k $(k\geq 4)$	$x^2 y + y^{k-1} + z^2 = 0$	2	$D_{k,2}$
	or		
	$x^2 y + xz^2 + y^2 = 0$	2	$D^*_{5,2}$
	or		
	$x^l y + xy^2 + z^2 = 0 \quad (k = 2l, l \geq 3)$,	l	$D_{k,l}$
	or		
	$x^l y + xz^2 + y^2 = 0 \quad (k = 2l + 1, l \geq 3)$	l	$D_{k,l}$
E_6	$x^2 z + y^3 + z^2 = 0$	2	E_6
E_7	$x^3 y + y^3 + z^2 = 0$	3	E_7

Proof It is just classifying the simple singularities again by choosing all the coordinate transformations in $\mathcal{R}_\Sigma$. To see the flavor of the computation, we only give the beginning of it.

All the functions having x-axis Σ in their zero sets and the origin in their singular loci form the ideal $(y, z)\mathfrak{m}$ of $\mathcal{O}$. The group $\mathcal{R}_\Sigma$ acts on $(y, z)\mathfrak{m}$. We consider the classification of germs in $(y, z)\mathfrak{m}$ under the equivalence defined by $\mathcal{R}_\Sigma$. For any $h \in (y, z)\mathfrak{m}$, we consider the 2-jet of h:

$$j^2 h = 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2, \quad a_{ij} = a_{ji}$$

Let r be the rank of the coefficient matrix $A := (a_{ij})$ of j^2h.

If $r = 3$, then $(a_{12}, a_{13}) \neq 0$, by exchanging y and z, we can assume $a_{12} \neq 0$. Then

$$j^2h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} xy + az^2 \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} xy + z^2 \overset{\mathcal{R}}{\leftrightsquigarrow} A_1$$

(we still use x, y, z to denote the new coordinates).

If $r = 2$, there are two cases to be treated.

First case: $(a_{12}, a_{13}) \neq 0$. By exchanging y and z, we can assume $a_{12} \neq 0$. Then $j^2h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} xy$. We consider higher order jet of h. We find that $j^{k+1}h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} xy + az^{k+1}$ which, by theorem 1.2.4, is $(k+1)$-$\mathcal{R}_\Sigma$-determined in $(y, z)\mathfrak{m}$, $\mathcal{R}_\Sigma$-equivalent to $xy + z^{k+1}$, which in turn is $\mathcal{R}$-equivalent to A_k, if $a \neq 0$.

Second case: $(a_{12}, a_{13}) = 0$. Since $r = 2$, $j^2h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} yz$. Consider the 3-jet of h:

$$j^3h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} yz + a_1x^2y + b_1x^2z$$

- $a_1b_1 \neq 0$: $j^3h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} yz + x^2y + x^2z \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} x^2y + y^2 + z^2$, which is 4-$\mathcal{R}_\Sigma$-determined in $(y, z)\mathfrak{m}$, and $\mathcal{R}$-equivalent to A_3.

- $a_1b_1 = 0, (a_1, b_1) \neq 0$: by exchanging y and z, we can assume $j^3h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} yz + x^2y$ which is not finitely $\mathcal{R}_\Sigma$-determined in $(y, z)\mathfrak{m}$. If we consider higher order jet step by step, we get

$$j^{k+1}h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} yz + x^2y + ax^kz \quad (k \geq 3)$$

  which, if $a \neq 0$, is $(k+1)$-$\mathcal{R}_\Sigma$-determined in $(y, z)\mathfrak{m}$, $\mathcal{R}_\Sigma$-equivalent to $yz + x^2y + x^kz$ which is $\mathcal{R}$-equivalent to A_{k+1}.

- $(a_1, b_1) = 0$: $j^3h \overset{\mathcal{R}_\Sigma}{\leftrightsquigarrow} yz$, consider the higher order jet of h.

If $r = 1$, we only find those $\mathcal{R}$-simple germs that are $\mathcal{R}$-equivalent to $D_k \quad (k \geq 4)$, E_6 or E_7.

E_8 does not appear in the list. An intuitive reason for this is that one can bring the 4-jet of the E_k $(k \geq 7)$ singularities into the normal form:

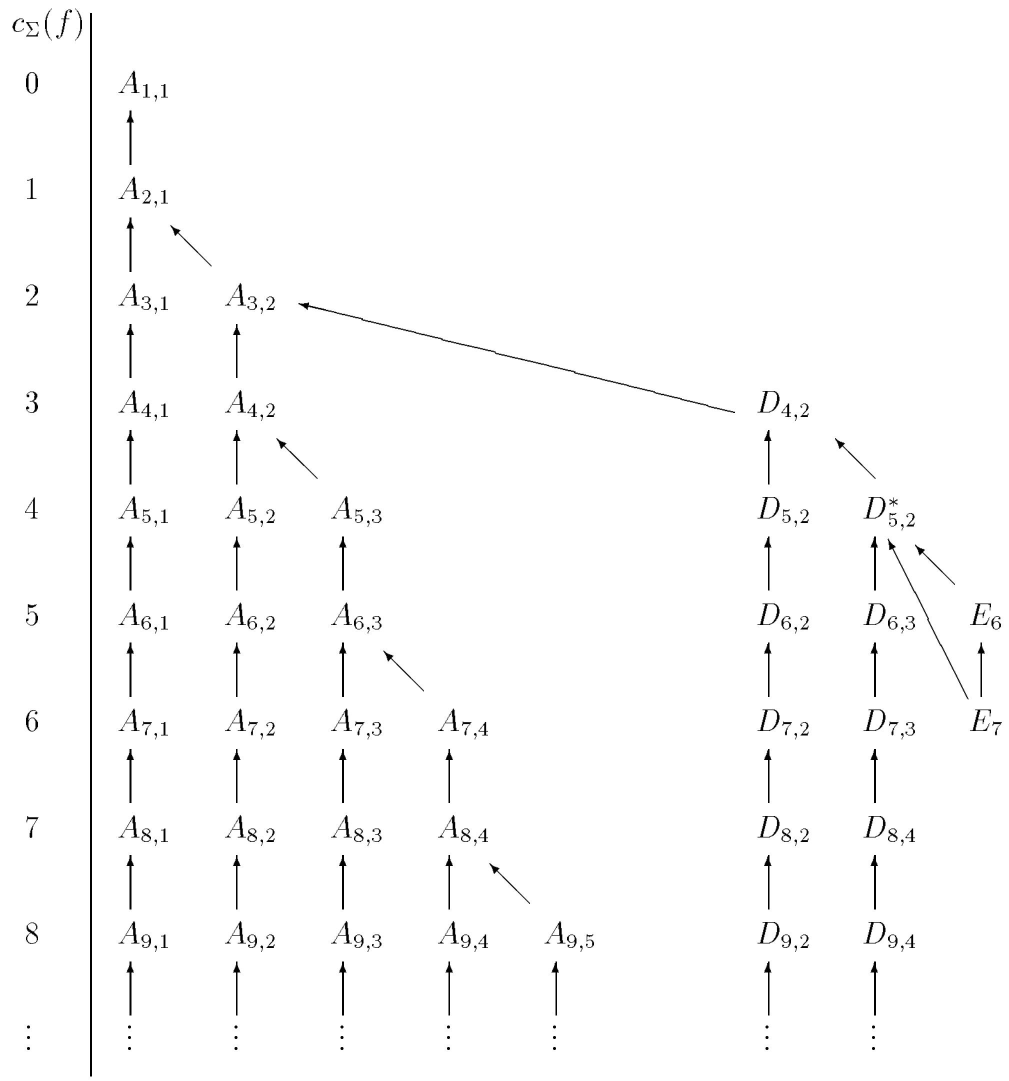
$$z^2 + y^3 + ax^3y.$$

In case $a \neq 0$, one has E_7, otherwise, there is no term x^4, so one can never get E_8.

One can easily calculate the $\lambda(h)$: $\lambda(h) = 1$, l or $\left[\frac{k+1}{2}\right]$ when h defines an A_k surface, $\lambda(h) = 2$ or $\left[\frac{k}{2}\right]$ when h defines a D_k surface, and so on. $\square$

1.4.5 The classification tree In order to give an outline of the classification we include the the classification tree. Note that this table contains only some of the adjacencies.

The Classification Tree



1.4.6 Remarks 1) There is only one class of smooth curves on A_1 (see also example 1 in §1.1.5), A_2, D_4, E_6 and E_7 surface. There exists two classes of smooth curves on D_5 surface with the same $\lambda = 2$. In §3.1, we will use the Milnor number of the stable germs to tell the differences between the two classes of smooth curves on D_5;

2) There exist $\left[\frac{k+1}{2}\right]$ classes of smooth curves on surfaces with A_k $(k \geq 3)$ type isolated singularties. On D_k $(k \geq 6)$ surfaces there are two classes of smooth curves. The number λ tells the differences of the smooth curves on these surfaces.

3) As a corollary of our calculation in the classification, we know the non-existence of smooth curve on surface with E_8 type isolated singularity.

4) All the normal forms of the germs in table 1 are also $\mathcal{R}_\Sigma$-simple. And all the $\mathcal{R}_\Sigma$ simple germs with isolated singularities in $\mathfrak{m}(y,z)$ are contained in table 1. One may expect that the codimension of the germs under the $\mathcal{R}_\Sigma$ action would depend on Σ, this in fact is not the case:

$$c_\Sigma(h) := \dim_{\mathbb{C}} \frac{(y,z)\mathfrak{m}}{\tau_{\mathfrak{g}}(h)}$$

does not depend on the position of Σ and X. This is a surprise for us.

5) It should be interesting to classify smooth curves on non-simple surfaces.

1.4.7 About $\lambda(h)$ For $h \in (y,z)\mathfrak{m}$ with isolated singularity at 0, there exists a deformation of h in $(y,z)\mathfrak{m}$ such that for generic parameters, the deformed function has only A_1 type critical points. Geometrically, $\lambda(h)$ is the intersection multiplicity of Σ with the critical locus of h. Is $\lambda(h)$ a constant with respect to the deformation? The answer is negative in general.

Let $\lambda'(X)$ denote the number of A_1 points sitting on Σ of a generic deformation of X. A calculation shows that $\lambda'(A_{k,1}) = 1$, $\lambda'(A_{2l+s,l}) = 1$, $\lambda'(A_{2l-1,l}) = l$, $\lambda'(D_{k,2}) = 2$, $\lambda'(D^*_{5,2}) = 1$, $\lambda'(D_{2l,l}) = l$, $\lambda'(D_{2l+1,l}) = 1$, $\lambda'(E_6) = 2$, $\lambda'(E_7) = 3$.

We observe that $\lambda(h)$ is the maximal number of A_1 points of h_s sitting on Σ, where h_s is a generic deformation of h in $(y,z)\mathfrak{m}$.

1.5 Classification

Classification of singularities is one of the central topics in singularity theory. Given an equivalence relation on some space of function (map) germs, one wants to know the representatives of the equivalence classes (or the normal forms of the germs in the classes). Under right equivalence and right-left equivalence, Siersma([S1] and [S2]) has classified the function germs with codimension not greater than 8. At the same time, Arnol'd also gave this classification and extended it later (see *e.g.* [AGV]). Among the normal forms are those without parameter, called simple germs which are important singularities and enjoy many interesting properties (see [Du]). Dimca[Di] and Lyashiko[Ly] have studied simple function germs with isolated singularities on hypersurfaces (see §3.3.6 for a quotation), and obtained lists of classifications. Tibăr[Ti] has obtained some normal forms of function germs on a hypersurface with $A(k)$ type non-isolated singularities. M. Zaharia [Za] has studied simple function germs with isolated singularities on arrangements of hyperplanes. Goryunov[Go] has classified functions on space curves.

When the space is smooth, Siersma has found all the simple germs with singular locus a smooth curve in [S3], and a plane curve with A_1 singularity in [S4]. A. Zaharia [Z1] has found all simple germs when the singular locus is a smooth manifold of dimension 2 or codimension 2.

The purpose of this section is to generalize a list of Dimca-Lyashiko to the non-isolated case. That is to classify simple functions with a smooth curve Σ as the singular locus, called line singularities, on X when X is A_k, D_k, E_6, or E_7 type two dimensional hypersurface singularity. The smooth curves on these surfaces have been described in §1.4. Especially when $\lambda(\Sigma X)$ is minimal, we classify all the simple germs on A_k surfaces in $\mathbb{C}^3$. We also consider the classifications for functions with singular locus a smooth curve on (higher dimensional) hypersurfaces with simple singularities.

It turns out that there exist no simple germs with singular locus a line Σ on a A_k $(k > 4)$ surface X_k when $(k > 4)$ and $1 < \lambda(\Sigma X_k) < \left[\frac{k+1}{2}\right]$. We list the germs with minimal codimensions in these cases. They will be used in chapter 3 to determine the homotopy type of the Milnor fibres.

1.5.1 We consider the right equivalence relation defined by $\mathcal{R}_{\Sigma X}$ on germs in $\int \mathfrak{g}$ (as functions on X). A germ $f \in \int \mathfrak{g}$ is $\mathcal{R}_{\Sigma X}-$*stable* (or stable if no confusion can be caused by this) if $c(f) = 0$. We call f to be $\mathcal{R}_{\Sigma X}-$*simple* (or simple if no confusion can be caused by this) if $c(f) < \infty$ and any deformation of f

$$\tilde{f} : (\mathbb{C}^{n+1} \times \mathbb{C}^s, 0) \longrightarrow (\mathbb{C}, 0)$$

cuts only a finite number of $\mathcal{R}_{\Sigma X}-$orbits of $\int \mathfrak{g}$ when u is sufficiently small, where $\tilde{f}(-, u) \in \int \mathfrak{g}$, and $\tilde{f}(z, 0) = f(z)$.

1.5.2 In the computations, the following lemma is used time after time. Even though the statement is standard, we still like to state it in our version. The proof of it is very similar to that of [S2](2.2) and (2.8), we omit it.

Lemma Let $f_t = f + t\phi \in \int_X \mathfrak{g}$ be a one parameter family of germs defined on $(X, 0)$ for t in a connected domain $I \subset \mathbb{C}$. Let $f_t \in \int_X \mathfrak{g}$ for all $t \in I$, $t_0 \in I$.

1) If $\phi \in J_{\Sigma X}(f_t) + \mathfrak{h}$, then f_t is $\mathcal{R}_{\Sigma X}$-equivalent to f_{t_0} for all $t \in I$;

2) If $\phi \notin J_{\Sigma X}(f_t) + \mathfrak{h}$ for every $t \in I$, then for any $t_0 \in I$, there exists an $\epsilon > 0$ such that whenever $0 <\mid t - t_0 \mid< \epsilon$, f_t is not $\mathcal{R}_{\Sigma X}$-equivalent to f_{t_0}. Namely f_t is not $\mathcal{R}_{\Sigma X}-$simple for any $t \in I$.$\square$

1.5.3 Theorem Let $X_k \subset \mathbb{C}^3$ be an A_k singularity defined by $h = xy + z^{k+1} = 0$ and $\mathfrak{g} = (y, z)$, that is, $\lambda(\Sigma X)$ is minimal. Then all $\mathcal{R}_{\Sigma X}$-simple germs in $\int \mathfrak{g}$ on X_1 are contained in table 2, and all $\mathcal{R}_{\Sigma X}$-simple germs in $\int \mathfrak{g}$ on X_k $(k > 1)$ are contained in table 3, where $d(f)$ is the determinacy degree of germ f, $\chi(F)$ is the Euler characteristic of the Milnor fibre F of f (see §3.2 or §3.3), and $\mu(f)$ is the Milnor number of f, that is, the number of circles in the bouquet decomposition of F.

Note that all the germs are weighted homogeneous. The notations D_∞ and $J_{k,\infty}$ has been used by Siersma[S3] where D_∞ and $J_{1,\infty}$ are equivalent. In our case $A_1 J_{1,\infty}$ and $A_1 D_\infty$ are not equivalent.

Table 2 : Non-Isolated Simple Germs on $A_{1,1}$ Surface

Name	Normal form of f	weights w_0, w_1, w_2	$c(f)$	$d(f)$	$j(f)$	$\chi(F)$	$\mu(f)$
$A_1 A_\infty$	y	1,1,1	0	1	0	1	0
$A_1 A_l$	$y^{l+1} + z^2$ $(l \geq 1)$	$2l, 2, 2(l+1)$	l	$l+1$	$l+1$	$-l-1$	$l+2$
$A_1 J_{l,\infty}$	$yz + x^l z^2$ $(l \geq 1)$	$1, 2l+1, l+1$	$l+1$	$l+2$	$2l+2$	$-3l-2$	$3l+3$
$A_1 D_\infty$	$y^2 + xz^2$	$2, 4, 3$	3	3	5	-6	7

Table 3 : Non-Isolated Simple Germs on $A_{k,1}$ Surfaces

Name	Normal form of f	Type of X	$c(f)$	$d(f)$	$j(f)$	$\chi(F)$	$\mu(f)$
A_kA_∞	$y+z^2$	$A_{k,1}$	0	1	1	-1	2
A_kA_l	$y^{l+1}+z^2$ $(l\geq 1)$	$A_{k,1}$	l	$l+1$	$l+1$	$-l-1$	$l+2$
$A_2J_{l,\infty}$	$y+x^lz^2$ $(l\geq 1)$	$A_{2,1}$	l	$l+2$	$2l+1$	$-3l-1$	$3l+2$
A_3D_∞	$y+xz^2$	$A_{3,1}$	1	3	4	-5	6

Remark For the computations of $\chi(F)$, we have two ways: using the A'Campo type formula (see [A'C] and [Ti3]) by resolutions of the singularities (see §3.3), and using the formula of §3.2 and *Singular*, a computer program developed by G.-M. Greuel, G Pfister and H. Schoenemann[GPS]. In fact, we prove in §3.1.13 that the Milnor fibre of a function f with singular locus a line Σ on an A_k surface X_k is homotopy equivalent to a bouquet of circles when $\lambda(\Sigma X)$ is minimal and $j(f)<\infty$. Then one can calculate the Milnor number $\mu(f)=1-\chi(F)$, or use the results in §3.1.13. (cf. §3.3.7 for more general statement).

Proof Since $\mathfrak{h}=(xy+z^{k+1})$ defines a weighted homogeneous *icis*, by [W] or [BR], we know that D_X is generated as $\mathcal{O}$-module by (see the example in §1.1.1)

$$\begin{aligned}
\xi_0 &= kx\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}, \quad \xi_1 = x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}, \quad \xi_2 = (k+1)z^k\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\\
\xi_3 &= (k+1)z^k\frac{\partial}{\partial y}-x\frac{\partial}{\partial z}, \quad \xi_4 = h\frac{\partial}{\partial x}, \qquad \xi_5 = h\frac{\partial}{\partial y}, \quad \xi_6 = h\frac{\partial}{\partial z}.
\end{aligned}$$

Hence ideal $J_{\Sigma X}(f)+\mathfrak{h}$ is generated by $\xi_0(f),\xi_1(f),\xi_2(f),y\xi_3(f),z\xi_3(f)$, and h. Note that $\int\mathfrak{g}=(y,z^2)$ and

$$\mathfrak{m}^{l+2}\cap\int\mathfrak{g}=\mathfrak{m}^{l+2}\cap(y,z^2)=\mathfrak{m}^l(\mathfrak{m}^2\cap(y,z^2))=\mathfrak{m}^l(\mathfrak{m}^2\cap\int\mathfrak{g}).$$

So the determinacy condition in theorem 1.2.2 is

$$\mathfrak{m}^{l+1}M_2\subset\mathfrak{m}J_{\Sigma X}(f)+\mathfrak{h}+\mathfrak{m}^{l+2}M_2\Longrightarrow f \text{ is } (l+2)-\text{determined in } \int_X\mathfrak{g}.$$

We only give the detailed classification of the germs on A_1 singularity here. A similar program will do the job for germs on $X_k (k\geq 2)$ and gives table 3.

It is easy to compute, for every germ f in Table 2 and Table 3, $\frac{\int_X \mathfrak{g}}{J_{\Sigma X}(f)+\mathfrak{h}}$, the weights, $c(f)$, $j(f)$ and the degree of determinacy. Notice that all $\phi \in \mathcal{R}_{\Sigma X}$ should take the form (in local coordinates)

$$j^1\phi : \begin{cases} x' \circ \phi = & a_{00}x + \quad a_{01}y \quad + a_{02}z \\ y' \circ \phi = & \qquad\qquad a_{11}y \\ z' \circ \phi = & \qquad\qquad a_{21}y \quad + a_{22}z \end{cases} \tag{1.5.3.1}$$

such that $\phi^*(h) = (h)$, namely $h \circ \phi = uh$, where u is a unit.

Our proof consists of four steps.

Step 1: Let $f \in \int \mathfrak{g} = (y, z^2)$. Then $f = ay + bz^2$ for $a, b \in \mathcal{O}$. Take one jet,

$$j^1 f = a(0)y.$$

If $a(0) \neq 0$, let

$$\phi : \begin{cases} x = & a(0)x' \\ y = & \qquad \frac{1}{a(0)}y' \\ z = & \qquad\qquad z' \end{cases}$$

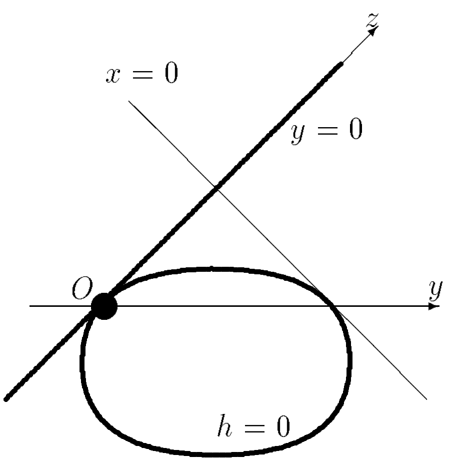


Figure 1

then

$$\begin{aligned} h \circ \phi = & \; x'y' + z'^2, \\ f \circ \phi = & \; y' + \text{ higher order terms,} \end{aligned}$$

which is 1-determined (see §1.5.4) . Hence

$$f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} A_1A_\infty \quad \text{(see Figure 1).}$$

If $j^1 f = 0$, we go to step 2.

Step 2

Let $j^2 f = ay^2 + byz + cz^2$.

Case I The rank r of the matrix

$$A = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$$

is 2, then $\Delta = 4ac - b^2 \neq 0$:

i) If $c \neq 0$, then

$$\phi : \begin{cases} x' = & cx \\ y' = & \quad y \\ z' = & \quad\quad \sqrt{c}z \end{cases}$$

makes $h \circ \phi^{-1} = c^{-1}(x'y' + z'^2)$, and

$j^2 f \circ \phi^{-1} = a'y'^2 + b'y'z' + z'^2 = (a' - \frac{b'^2}{4})y'^2 + (z' + \frac{b'}{2}y)^2$.

Let

$$\psi : \begin{cases} x'' = & x' - \frac{b'^2}{4}y' - b'z' \\ y'' = & y' \\ z'' = & \frac{b'}{2}y' + z' \end{cases}$$

Then we have

$$h \circ \phi^{-1} \circ \psi^{-1} = c^{-1}(x''y'' + z''^2),$$

and

$$j^2 f \circ \phi^{-1} \circ \psi^{-1} = (a' - \frac{b'^2}{4})y''^2 + z''^2 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} A_1 A_1 \tag{1.5.3.2}$$

since $\Delta \neq 0$ (see Figure 2).

ii) If $c = 0$, then $b \neq 0$ since $\Delta \neq 0$. Hence

$$j^2 f = y(ay + bz) \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} yz \tag{1.5.3.3}$$

which is not finitely determined (see Figure 3). We shall consider higher order jet of f in step 3.

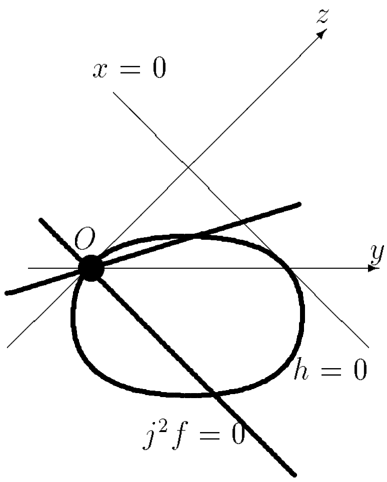


Figure 2

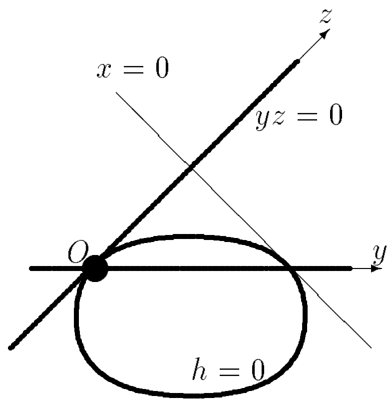


Figure 3

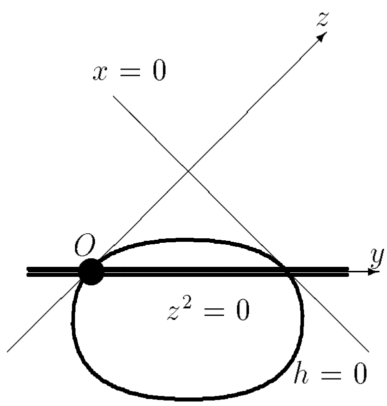


Figure 4

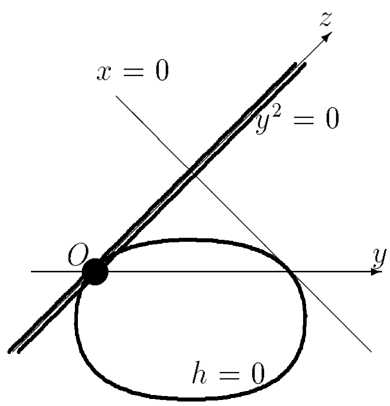


Figure 5

Case II $r = 1$, namely $\Delta = 0$, so

$$j^2 f = (\alpha y + \beta z)^2.$$

i) $\beta \neq 0$, let

$$\phi : \begin{cases} x' = & \beta^2 x - & \alpha^2 y & -2\alpha\beta z \\ y' = & & y & \\ z' = & & \alpha y & +\beta z \end{cases}$$

Then

$$h \circ \phi^{-1} = \beta^{-2}(x'y' + z'^2)$$

and

$$j^2 f \circ \phi^{-1} = z'^2 \tag{1.5.3.4}$$

which is not finitely determined (see Figure 4).

ii) $\beta = 0, \alpha \neq 0$ since $r = 1$, and

$$j^2 f \quad \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} \quad y^2 \tag{1.5.3.5}$$

which is not finitely determined (see Figure 5).

Step 3 We consider three cases:

Case I $j^2 f = z^2$.

On X we have

$$f = z^2 + axz^2 + by^3 + b_2 y^2 z + b_3 yz^2 + b_4 z^3 + \text{ higher order terms.}$$

First, z^2 will kill all the degree 3 terms with z^2. Let $z' = z + \frac{1}{2} b_2 y^2$. Choosing x' properly will kill $y^2 z$. Hence f is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_1 = z^2 + by^3 + \text{ higher order terms.}$$

If $b \neq 0$, then f_1 is $\mathcal{R}_{\Sigma X}$-equivalent to $f_1 = z^2 + y^3$ which is 3-determined.

In general, if $j^{l-1} f = z^2$, we know that $j^l f$ is $\mathcal{R}_{\Sigma X}$-equivalent to either

$$z^2 + y^l$$

which is l-determined, or z^2 which is not finitely determined.

Case II $j^2 f = yz$.

On X we can write

$$f = yz + axz^2 + b_1 y^3 + b_2 y^2 z + b_3 xy^2 + b_4 xyz + \text{ higher order terms}$$

which is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_1 = yz + axz^2 + \text{ higher order terms.}$$

If $a \neq 0$, f_1 is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_2 = yz + xz^2 + \text{ higher order terms.}$$

Obviously, $f_3 = yz + xz^2$ is 3-determined. Hence when $a \neq 0$, f is $\mathcal{R}_{\Sigma X}$-equivalent to $yz + xz^2$.

If $j^{l-1}f = yz$, $j^l f$ is $\mathcal{R}_{\Sigma X}$-equivalent to either

$$yz + x^{l-2}z^2$$

which is l-determined, or yz which is not finitely determined.

Case III $j^2 f = y^2$.

On X we can let

$$f = y^2 + axz^2 + b_1y^3 + b_2y^2z + b_3yz^2 + bz^3 + \text{ higher order terms}$$

which, on X, is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_1 = y^2 + axz^2 + bz^3 + \text{ higher order terms.}$$

i) $a \neq 0$,

$$f_1 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + xz^2 + tz^3$$

which is 3-determined, and since $z^3 \in J_{\Sigma X}(f_1) + \mathfrak{h}$, by lemma 1.5.2, we know that

$$f_1 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + xz^2$$

which is 3-determined and simple.

ii) $a = 0, b \neq 0$, we consider higher order jet of f_1:

$$f_1 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + z^3 + tx^2z^2$$

which is 4-determined (when $t \neq 0, \frac{1}{4}$). Since $x^2z^2 \notin J_{\Sigma X}(f_1) + \mathfrak{h}$, by lemma 1.5.2, we know that $y^2 + z^3 + tx^2z^2$ is not simple.

iii) $a = 0, b = 0$, we consider higher order jet. In general

$$j^l f_1 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + ax^{l-2}z^2 + bx^{l-3}z^3.$$

If $a \neq 0$, then

$$f_1 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + x^{l-2}z^2 + tx^{l-3}z^3$$

which is $l-$determined with $x^{l-3}z^3 \in J_{\Sigma X}(f_1) + \mathfrak{h}$ by lemma 1.5.2, we know that

$$f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + x^{l-2}z^2.$$

If $a = 0, b \neq 0$, then

$$j^l f_1 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + x^{l-3}z^3$$

which is not finitely determined, and

$$j^{l+1} f_1 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + x^{l-3}z^3 + tx^{l-1}z^2,$$

when $t \neq 0$, which is $l+1$-determined, but not simple by lemma 1.5.2 since $x^{l-3}z^3 \notin J_{\Sigma X}(j^{l+1}f_1) + \mathfrak{h}$.

Hence for $l \geq 4$, since

$$y^2 + x^{l-3}z^3 + tx^{l-2}z^2 \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + sx^{l-3}z^3 + x^{l-2}z^2$$

is not simple and $y^2 + x^{l-2}z^2$ is in the boundary of a non-simple germ. So it is not simple.

Step 4 $j^2 f = 0$.

On X we can write

$$f = axz^2 + b_1y^3 + b_2y^2z + b_3yz^2 + b_4z^3 + \text{ higher order terms.}$$

We first assume that $a \neq 0$ and consider the following cases.

1) If $b_1b_2 \neq 0, b_3 = b_4 = 0$, then f is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_1 = xz^2 + y^3 + ty^2z$$

which is 3-determined with $c(f) = 5$. Since $y^2z \notin J_{\Sigma X}(f) + \mathfrak{h}$, by lemma 1.5.2, f is not simple.

2) If $b_1b_3 \neq 0, b_2 = b_4 = 0$, then f is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_2 = xz^2 + ty^3 + yz^2 \quad (t \neq 0, -\frac{1}{4}) \quad or \quad xz^2 + syz^2 + y^3 \quad (s^2 \neq -4)$$

which is 3-determined with $c(f) = 5$. Since $y^3, yz^2 \notin J_{\Sigma X}(f) + \mathfrak{h}$, by lemma 1.5.2, f is not simple.

3) If $b_1b_4 \neq 0, b_2 = b_3 = 0$, then f is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_3 = xz^2 + sy^3 + z^3$$

which is 3-determined with $c(f) = 5$. Since $y^3 \notin J_{\Sigma X}(f) + \mathfrak{h}$, by lemma 1.5.2, f is not simple.

4) If $b_2b_3 \neq 0, b_1 = b_4 = 0$, then f is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_4 = xz^2 + y^2z + tyz^2 \quad (t \neq 0) \quad or \quad sxz^2 + y^2z + yz^2$$

which is 3-determined with $c(f) = 5$. Since $yz^2, xz^2 \notin J_{\Sigma X}(f) + \mathfrak{h}$, by lemma 1.5.2, f is not simple.

5) If $b_2b_4 \neq 0, b_1 = b_3 = 0$, then f is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_5 = xz^2 + ty^2z + z^3 \quad (t \neq 0)$$

which is 3-determined with $c(f) = 5$. Since $y^2z \notin J_{\Sigma X}(f) + \mathfrak{h}$, by lemma 1.5.2, f is not simple.

6) If $b_3b_4 \neq 0, b_1 = b_2 = 0$, then f is $\mathcal{R}_{\Sigma X}$-equivalent to

$$f_6 = xz^2 + tyz^2 + z^3$$

which is not finitely determined. Since

$$f_7 = xz^2 + tyz^2 + sy^3 + z^3$$

is 3-determined ($st \neq 0$) and not simple because yz^2 and $y^3 \notin J_{\Sigma X}(f) + \mathfrak{h}$. Since f_6 is on the boundary of f_7, f_6 is not simple. Hence for $a_1 \neq 0$, there is no simple germ.

If $a_1 = 0$, then

$$j^3 f = b_1y^3 + b_2y^2z + b_3yz^2 + b_4z^3$$

is $\mathcal{R}_{\Sigma X}-$equivalent to one of the following forms

$$0, y^3, y^2z, yz^2, z^3, y^2z + z^3, y^2z + yz^2, yz^2 + z^3, y^2z + tyz^2 + z^3.$$

None of them is finitely determined. If we consider the higher order jet of f, no simple germs. □

1.5.4 In this and the next subsections, we consider germs with line singularities on some (higher dimensional) hypersufaces with A_k ($k \geq 1$), D_k ($k \geq 4$), E_6 or E_7 type singularities. Let $\Sigma = x -$ axis in $\mathbb{C}^{n+1}$, defined by $\mathfrak{g} = (y_1, \dots, y_n)(n \geq 2)$, and $X_k \subset \mathbb{C}^{n+1}$ be a hypersurface with isolated singularity at 0 of type A_k, defined by

$$h_k = xy_1 + y_2^2 + \cdots + y_{n-1}^2 + y_n^{k+1} = 0.$$

Then $\int \mathfrak{g} = (y_1) + (y_2, \dots, y_n)^2$. And for any $f \in \int \mathfrak{g}$, we have

$$f = ay_1 + \sum_{i,j=2}^{n} a_{ij}y_iy_j \qquad (a, a_{ij} \in \mathcal{O}).$$

Theorem Let $a(0) \neq 0$.

1) On X_1, $f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y_1$ which is 1-determined and stable. And there are no other simple germs when $n > 2$;

2) On $X_k(k > 1)$, $f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y_1 + y_n^2$ which is 2-determined and stable if $a_{nn}(0) \neq 0$;

3) On $X_k(k > 1)$, if $a_{nn}(0) = 0$, then $j^2 f$ is finitely $\mathcal{R}_{\Sigma X}$-determined if and only if $\alpha_n := \sum[a_{jn}(0)]^2 \neq 0$. And when $\alpha_n \neq 0$, we can change the order of $y_2, \dots, y_{n-1}$ such that

$$j^2 f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y_1 + y_2y_n.$$

Proof For the case of $n = 2$ see Theorem 1.5.3. We assume $n \geq 3$. Note that D_{X_k} can be generated, as $\mathcal{O}_{n+1}$-module, by

$$\begin{aligned}
\xi_0 =& 2kx\frac{\partial}{\partial x} + 2y_1\frac{\partial}{\partial y_1} + (k+1)y_2\frac{\partial}{\partial y_2} + \cdots + (k+1)y_{n-1}\frac{\partial}{\partial y_{n-1}} + 2y_n\frac{\partial}{\partial y_n},\\
\xi_{01} =& x\frac{\partial}{\partial x} - y_1\frac{\partial}{\partial y_1},\\
\xi_{0\mu} =& 2y_\mu\frac{\partial}{\partial x} - y_1\frac{\partial}{\partial y_\mu}, \qquad \mu = 2, \ldots, n-1,\\
\xi_{0n} =& (k+1)y_n^k\frac{\partial}{\partial x} - y_1\frac{\partial}{\partial y_n},\\
\xi_{st} =& 2y_t\frac{\partial}{\partial y_s} - 2y_s\frac{\partial}{\partial y_t}, \qquad 2 \leq s < t \leq n-1,\\
\xi_{\nu n} =& (k+1)y_n^k\frac{\partial}{\partial y_\nu} - y_\nu\frac{\partial}{\partial y_n}, \qquad \nu = 2, \ldots, n-1,
\end{aligned}$$

and

$$\begin{aligned}
\xi_{1j} =& 2y_j\frac{\partial}{\partial y_1} - x\frac{\partial}{\partial y_j}, \qquad j = 2, \ldots, n-1,\\
\xi_{1n} =& (k+1)y_n^k\frac{\partial}{\partial y_1} - x\frac{\partial}{\partial y_n},
\end{aligned}$$

where the last $n-1$ derivations $\xi_{1\mu}(\mu = 2, \ldots, n)$ do not preserve $\mathfrak{g}$, but the $y_i\xi_{1\mu}$'s do.

1) It is easy to check that $f = y_1$ is 2-determined and codimension 0 in $\mathfrak{g}$. A similar calculation to the proof of 2) will prove y_1 is 1-determined and stable.

2) If $a(0) \neq 0$, $j^1 f = a(0)y_1 \overset{\mathcal{R}_{\Sigma X_k}}{\leftrightsquigarrow} y_1$ which is not finitely determined in $\int \mathfrak{g}$. It is obvious that

$$j^2 f \overset{\mathcal{R}_{\Sigma X_k}}{\leftrightsquigarrow} y_1 + \sum_{i,j=2}^{n} a_{ij}(0)y_iy_j.$$

In the following, we denote $a_{ij}(0)$ by a_{ij} again. Let

$$g = j^2 f = y_1 + Q \quad \text{ with } Q = \sum_{i,j=2}^{n} a_{ij}y_iy_j.$$

We are going to prove that g is 2-determined and codimension 0 under the condition $a_{nn} \neq 0$. This is equivalent to expressing y_1 and all $y_iy_j(i, j = 2, \ldots, n)$ in terms of the $\xi_0(g)$ and $\xi_{kl}(g)$.

In fact, we have $\xi_{01}(g) = -y_1$ and

$$\xi_0(g) + \xi_{01}(g) = (k+1)\sum_{j=2}^{n-1} \frac{\partial Q}{\partial y_j} + 2y_n \frac{\partial Q}{\partial y_n} \tag{1.5.4.1}$$

and

$$\xi_{1j}(g) = 2y_j - x\frac{\partial Q}{\partial y_j}, \quad j = 2, \ldots, n-1.$$

Then

$$y_\lambda \xi_{1j}(g) = 2y_\lambda y_j - xy_\lambda \frac{\partial Q}{\partial y_j} = 2\sum_{i=2}^{n} (\delta_{ij} - a_{ij}x) y_i y_\lambda$$

for $j = 2, \ldots, n-1$ and $\lambda = 2, \ldots, n$.

Namely we get $n-2$ systems of equations

$$\sum_{i=2}^{n} (\delta_{ij} - a_{ij}x) y_i y_\lambda \equiv 0 (\mod J_{\Sigma X_k}(f)), \quad \lambda = 2, \ldots, n \tag*{$(1.5.4.2)_j$}$$

For every $j = 2, \ldots, n-1$, we delete the first $j-2$ equations from $(1.5.4.2)_j$. The remained $n-j+1$ equations form a system of equations denoted by E_j^{n-j+1}. The union of E_j^{n-j+1} is a system of equations, denoted by E_n, with $\frac{n(n-1)}{2} - 1$ equations, and $\{y_j y_\lambda\}_{j=2,\ldots,n-1,\lambda=2,\ldots,n}$ as unknowns. The number of the unknowns is also $\frac{n(n-1)}{2} - 1$. The coefficient matrix M consists of $1 - a_{ii}x$ on the principal diagonal which are units in $\mathcal{O}$ and $a_{ij}x$ off the principal diagonal. Hence the determinant of the coefficient matrix is a unit in $\mathcal{O}$. Thus we can assume that

$$y_i y_j \equiv \lambda_{ij} x y_n^2 \quad (\mod J_{\Sigma X_k}(f)), \quad i = 2, \ldots, n-1, j = 2, \ldots, n.$$

From (1.5.4.1), we have

$$\left(a_{nn} + x\sum a_{ij}\lambda_{ij}\right) y_n^2 \equiv 0 \quad (\mod J_{\Sigma X_k}(f)).$$

Since $a_{nn} \neq 0, y_n^2 \equiv 0 \quad (\mod J_{\Sigma X_k}(f))$. From this we have

$$y_i y_j \equiv 0 \quad (\mod J_{\Sigma X_k}(f)), \quad i = 2, \ldots, n-1, j = 2, \ldots, n.$$

This proves that f is 2-determined and, by lemma 1.5.2, $\mathcal{R}_{\Sigma X_k}$-equivalent to $y_1 + y_n^2$ which is stable.

3) If we add another equation (see (1.5.4.1))

$$(k+1)\sum_{j=2}^{n-1} \frac{\partial Q}{\partial y_j} + 2y_n \frac{\partial Q}{\partial y_n} \equiv 0 \quad (\mod J_{\Sigma X_k}(f)) \tag{1.5.4.3}$$

to the system E_n as the last equation, we get a new system of equations E_n^e, which is homogeneous if we view xy_n^2 as the last unknown. The coefficient matrix M^e of E_n^e is obtained from M by adding the coefficients of xy_n^2 as the last colum and the coefficients of (1.5.4.3) as the last raw. $j^2 f$ is $\mathcal{R}_{\Sigma X}$-finitely determined in $\int \mathfrak{g}$ if and only if the determinant of M^e is a unit in $\mathcal{O}$. A calculation shows that

$$\det M^e \equiv \alpha_n(\bmod (x)\mathcal{O}_1). \qquad \square$$

1.5.5 Theorem Let $\Sigma = x -$ axis in $\mathbb{C}^{n+2}$, defined by $\mathfrak{g} = (y, z_1 \dots, z_n)$.

1) Let $X_k \subset \mathbb{C}^{n+2}$ be a hypersurface with isolated singularity at 0 of type D_k defined by

$$h_k = x^2 y + y^{k-1} + z_1^2 + \cdots + z_n^2 = 0 \quad (k \geq 4),$$

then

$$\int \mathfrak{g} = (y) + (z_1, \dots, z_n)^2 \tag{1.5.5.1}$$

and for any $f \in \int \mathfrak{g}$, we can write

$$f = ay + \sum_{i,j=1}^{n} a_{ij} z_i z_j \qquad (a, a_{ij} \in \mathcal{O}) \tag{1.5.5.2}$$

which is $\mathcal{R}_{\Sigma X_k}$-equivalent to y, 1-determined and stable if $a(0) \neq 0$. And if $n \geq 1$, there are no other simple germs.

2) Let Y be a hypersurface with isolated singualrity at 0 of type E_6 defined by

$$x^2 y + y^2 + z_1^3 + z_2^2 + \cdots + z_n^2 = 0,$$

then $\int \mathfrak{g}$ is the same as in (1.5.5.1), for any $f \in \int \mathfrak{g}$ we can write f as (1.5.5.2), f is $\mathcal{R}_{\Sigma Y}$-equivalent to $y + tz_1^2$ which is 2-determined not stable and not simple with $t \neq 0$ a parameter.

3) Let Z be a hypersurface with isolated singualrity at 0 of type E_7 defined by

$$x^3 y + y^3 + z_1^2 + \cdots + z_n^2 = 0,$$

then $\int \mathfrak{g}$ is the same as (1.5.5.1) and for any $f \in \int \mathfrak{g}$ we can write f as (1.5.5.2), which is $\mathcal{R}_{\Sigma Z}$-equivalent to y, 1-determined and stable. And if $n \geq 1$, then there are no other simple germs.

Proof One can prove this theorem in the same way as 1.5.4. $\square$

1.5.6 One may be interested in finding the simple germs on a A_k ($k \geq 3$) or a D_k ($k \geq 5$) surface X when $\lambda(\Sigma X)$ is not minimal.

Theorem 1) Let $\mathfrak{g} = (y, z)$, and X be a two dimensional hypersurface with A_k or D_k type singularity defined by :

$A_{2l-1,l}$: $x^l y + y^2 + z^2 = 0$,

$A_{2l,l}$: $x^l y + yz + xz^2 = 0$,

$D_{2l,l}$: $x^l y + xy^2 + z^2 = 0 \quad (l \geq 2)$,

or

$D_{2l+1,l}$: $x^l y + xz^2 + y^2 = 0 \quad (l \geq 2)$.

then germ $f = y : X \longrightarrow \mathbb{C}$ is stable. Note that $\lambda(\Sigma X_k) = \left[\frac{k+1}{2}\right]$;

2) For Σ defined by $y = z = 0$ on A_k ($k \geq 3$) type surface $X_k \subset \mathbb{C}^3$ with $1 < \lambda(\Sigma X_k) < \left[\frac{k+1}{2}\right]$, there does not exist stable germs in $\int(y, z) = (y, z^2)$. However table 4 gives the normal forms of germs with minimal codimension, where $t = \min\{l, s+1\} - 2$. $\square$

Table 4: Normal Forms of Germs with Minimal Codimension

Type of X_k	Equations	λ	Normal Forms of f	$c(f)$	$d(f)$	$j(f)$
$A_{5,2}$	$x^2y + x^2z^2 + yz = 0$	2	$y + bz^2$ $(b \neq 0, 1)$	1	2	3
$A_{6,2}$	$x^2y + x^3z^2 + yz = 0$	2	$y + bz^2$ $(b \neq 0, 1)$	1	2	3
$A_{7,2}$	$x^2y + x^4z^2 + yz = 0$	2	$y + bz^2$ $(b \neq 0)$	1	2	3
$A_{7,3}$	$x^3y + x^2z^2 + yz = 0$	3	$y + bz^2$ $(b \neq 0)$	1	2	3
$A_{8,2}$	$x^2y + x^5z^2 + yz = 0$	2	$y + bz^2$ $(b \neq 0)$	1	2	3
$A_{8,3}$	$x^3y + x^3z^2 + yz = 0$	3	$y + bz^2$ $(b \neq 0)$	2	3	5
$A_{k,l}$	$x^ly + x^{s+1}z^2 + yz = 0$ $(k = 2l + s \geq 9)$	l	$y + \sum_{i=0}^{t} b_i x^i z^2$ $(b_0 \neq 0)$	$t+1$	$t+2$	$2t+3$

Chapter 2

Conormal Modules and Deformations

2.1 Conormal Modules and Primitive Ideals

The conormal module of an ideal $\mathfrak{a}$ of a ring R is the $R/\mathfrak{a}$-module $M := \mathfrak{a}/\mathfrak{a}^2$. This module is very important both in algebra and in geometry and has been studied by many authors. Vasconcelos has studied when $\mathfrak{a}$ is generated by a regular sequence in [Va1], and the relationship between " $\mathfrak{a}$ is generated by a regular sequence" and "M has finite projective dimension" in [Va2]. If one assumes the former, the later is equivalent to "M is a free $R/\mathfrak{a}$-module". Higher conormal modules and syzygies have been studied by Simis and Vasconcelos in [Si] and [SV]. In his paper [Ku], Kunz has studied conditions that imply M to be torsion free.

However, when R is not regular, the most interesting conormal modules are the ones which are not free, not torsion free. Especially, no generator set of $\mathfrak{a}$ forms a regular sequence. Then the length (when it is finite) of the torsion part $T(M)$ of M, conditions on freeness of the torsion free module $N := M/T(M)$, and splitting of the exact sequence (2.1.1.1) are very interesting. Also there are some homology and cohomology groups T_i's and T^i's associated with M. In this section, we study mainly these questions in a certain geometric environment. For the purpose of this chapter, we pay special attention to the freeness of N. We also give a formula for the generators of the primitive ideal.

2.1.1 Let X be a reduced analytic space germ in $(\mathbb{C}^m, 0)$ defined by the radical ideal $\mathfrak{h}$ of $\mathcal{O}$. Let Σ be the germ of a subspace of X defined by the radical ideal $\mathfrak{g}$ of $\mathcal{O}$. Denote $\mathcal{O}_X = \frac{\mathcal{O}}{\mathfrak{h}}$, $\mathcal{O}_\Sigma = \frac{\mathcal{O}}{\mathfrak{g}}$, and $\bar{\mathfrak{g}} = \frac{\mathfrak{g}}{\mathfrak{h}}$. Remark that

$\bar{\mathfrak{g}}$ is also radical. Hence, being viewed as a subspace of X, Σ is also reduced defined by $\bar{\mathfrak{g}}$. The $\mathcal{O}_\Sigma$-module

$$M := \frac{\bar{\mathfrak{g}}}{\bar{\mathfrak{g}}^2} \cong \frac{\mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}}$$

is called the *conormal module (or bundle)* of $\bar{\mathfrak{g}}$.

Let M' be an $\mathcal{O}_\Sigma$-module, $S \subset \mathcal{O}_\Sigma$ be the set of all the non-zero divisors of $\mathcal{O}_\Sigma$, Q be the *total quotient ring* of $\mathcal{O}_\Sigma$ with denominator set S, and $M'_S := M' \otimes Q$, the *quotient module* of M'. There is a canonical map $i : M' \longrightarrow M'_S : i(m) = \frac{m}{1}$ for any $m \in M'$. The kernel $T(M') := \ker(i) = \{m \in M' \mid \exists s \in S, sm = 0\}$ is called the *torsion (part)* of M'. If $T(M') = 0$, we say that M' is *torsion free.* The factor module $M'/T(M')$ is torsion free. These fit into the exact sequences

$$0 \longrightarrow T(M') \longrightarrow M' \longrightarrow M'_S,$$

$$0 \longrightarrow T(M') \longrightarrow M' \longrightarrow M'/T(M') \longrightarrow 0 \qquad (2.1.1.1)$$

Note that $M' \otimes Q = 0$ if and only if $T(M') = M'$. If this is the case, M' is called a *torsion module.* If $M' \otimes Q$ is a free Q-module of rank r, we say that *M' has rank r.* M' has rank r if and only if there exists a free submodule $M_1 \subset M'$ of rank r such that M'/M_1 is a torsion module.

Let $\mathfrak{g} = (g_1, \ldots, g_n)$ and $\bar{\mathfrak{g}} = (\bar{g}_1, \ldots, \bar{g}_n) = \frac{\mathfrak{g}}{\mathfrak{h}}$. There is an exact sequence:

$$0 \longrightarrow \mathfrak{R} \xrightarrow{\alpha} \mathcal{O}_X^n \xrightarrow{g} \bar{\mathfrak{g}} \longrightarrow 0 \qquad (2.1.1.2)$$

of $\mathcal{O}_X$-modules, where $\mathfrak{R} := \ker(g)$, the module of relations among the $\bar{g}_i$'s. Let $\mathfrak{R}_0 \subset \mathfrak{R}$ be the submodule of $\mathfrak{R}$ generated by the trivial relations of $\bar{g}_i$'s: $\bar{g}_i e_j - \bar{g}_j e_i$, where e_i's are the canonical basis of $\mathcal{O}_X^n$. Then $\mathfrak{R}/\mathfrak{R}_0$ is an $\mathcal{O}_\Sigma$-module, denoted by $H_1(\bar{\mathfrak{g}}, \mathcal{O}_X)$, which is the first Koszul homology of the squence: $\bar{g}_1, \ldots, \bar{g}_n$.

Tensoring (2.1.1.2) with $\mathcal{O}_\Sigma$, we have a presentation of M:

$$H_1(\bar{\mathfrak{g}}, \mathcal{O}_X) \xrightarrow{\bar{\alpha}} \mathcal{O}_\Sigma^n \xrightarrow{\bar{g}} M \longrightarrow 0 \qquad (2.1.1.2)'$$

There is a complex, called the *Lichtenbaum-Schlessinger complex*, of $\mathcal{O}_\Sigma$-modules:

$$\mathbf{LS}: \quad 0 \longrightarrow H_1(\bar{\mathfrak{g}}, \mathcal{O}_X) \xrightarrow{\partial_2} \mathcal{O}_\Sigma^n \xrightarrow{\partial_1} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma \longrightarrow 0 \qquad (2.1.1.3)$$

where Ω_X^1 is the module of Kähler 1-forms on X, and the morphisms ∂_1 and ∂_2 are defined by:

$$\partial_1(\sum a_j e_j) = \sum a_j \delta \bar{g}_j, \text{ and } \partial_2([r]) = \sum r_j e_j,$$

where $\delta(\bar{b}) = db \otimes 1$ and $d : \mathcal{O}_X \longrightarrow \Omega^1_X$ is the differential of $\mathcal{O}_X$. Notice that $\bar{\alpha} = \partial_2$. Denote by $T_0(\Sigma X), T_1(\Sigma X)$ and $T_2(\Sigma X)$ the homology groups of (2.1.1.3).

Proposition 1) $T_0(\Sigma X) \cong \Omega^1_\Sigma$;
2) $T_1(\Sigma X) = 0$ if Σ is a line on an *icis* X;
3) $T_2(\Sigma X) = 0$ when Σ is strongly Cohen-Macaulay in X.

Proof By considering (2.1.1.2)′, (2.1.1.3) and the second exact sequence (1.1.7.1) together, one finds that $\delta \circ \bar{g} = \partial_1$. This proves 1). For 2), one will find that the second exact sequence is also exact on the left (see (2.1.6.2)) if Σ is a line on an *icis* .

The conclusion in 3) is the proposition 2.4 of Huneke[Hu]. We recall the definition of *strongly Cohen-Macaulay subscheme* from [Hu]. Let R be a Cohen-Macaulay local ring, $Y \subset X := \mathrm{Spec}(R)$ be defined by an ideal $\mathfrak{g} = (g_1, \dots, g_n)$ of R. If the ith Koszul homology groups $H_i(\mathfrak{g}; R)$ (associated to the sequence $g_1, \dots, g_n$) are either 0 or Cohen-Macaulay modules, and Y is generically a complete intersection, then we call Y a strongly Cohen-Macaulay subscheme in X. Y is generically a complete intersection means that $\mathfrak{g}_\mathfrak{p}$ is generated by a regular sequence for every minimal prime over ideal $\mathfrak{p}$ of $\mathfrak{g}$. □

Remark One can find more discussions about $T_2(\Sigma X)$ in [SV]. Especially, $T_2(\Sigma X)$ does not depend on the generators of $\bar{\mathfrak{g}}$. And $\bar{\mathfrak{g}}$ is called a *syzygetic ideal* if $T_2(\Sigma X) = 0$.

2.1.2 Let $\Sigma \subset X$ be the same as in §2.1.1. Denote $X_{\mathrm{reg}} := X \setminus X_{\mathrm{sing}}$. Assume Σ is a complete intersection in $\mathbb{C}^m$. Remark that $\Sigma \cap X_{\mathrm{reg}}$ is a local complete intersection in X_{reg}.

In the remainder of this section, we will stick to the following assumption on $\Sigma \subset X$ of §2.1.1:

$$\boxed{(\nabla): \quad X_{\mathrm{sing}} \subsetneqq \Sigma \quad \text{and} \quad \dim X_{\mathrm{sing}} < \dim \Sigma.}$$

Lemma If Σ is a complete intersection in $\mathbb{C}^m$, then

$$T(M) = T := \frac{\int \mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}}.$$

Proof Let U be an open neighborhood of 0 in $\mathbb{C}^m$ in which X and Σ are defined. Let $V = \Sigma \setminus (\Sigma_{\mathrm{sing}} \cup X_{\mathrm{sing}})$. Then V is an open dense subset of Σ since (∇). Then $T = 0$ on V since Σ is a reduced local complete intersection in X_{reg} (see [P2]). Hence $T \subset T(M)$.

For any $\bar{a} \in T(M)$, let $\bar{\lambda} \in \mathcal{O}_\Sigma$ be a non-zero divisor such that $\bar{\lambda}\bar{a} = 0$. By taking representatives we have $\lambda a \in \mathfrak{g}^2 + \mathfrak{h}$. So for any $\xi \in D_X$, we have $\xi(\lambda a) \equiv \lambda\xi(a) \pmod{\mathfrak{g}}$. But $\xi(\lambda a) \in \mathfrak{g}$, hence $\xi(a) \in \mathfrak{g}$. This tells us that $a \in \int \mathfrak{g}$. $\square$

2.1.3 Denote $N := \frac{\mathfrak{g}}{\int \mathfrak{g}}$. Under the assumption (∇), we have the exact sequence

$$0 \longrightarrow T(M) \longrightarrow M \longrightarrow N \longrightarrow 0 \tag{2.1.3.1}$$

Proposition Let Σ be a complete intersection in $\mathbb{C}^m$. If one can choose an $\mathcal{O}$-regular sequence $g_1, \dots, g_n$ which is the minimal generator set of $\mathfrak{g}$, such that

a) there exists an integer $0 \leq t \leq n$ such that the images of $g_1, \dots, g_t$ are in $T(M)$;

b) The germs $(g_{t+1})_z, \dots, (g_n)_z$ at $z \in X_{\text{reg}}$ form an $\mathcal{O}_{X,z}$-regular sequence,

then

1) $\int \mathfrak{g} = (g_1, \dots, g_t) + (g_{t+1}, \dots, g_n)^2$;

2) N is free $\mathcal{O}_\Sigma$-module of rank $n - t$, (2.1.3.1) splits and $M = N \oplus T(M)$.

Proof 1) If $f = a_1 g_1 + \dots + a_n g_n \in \mathfrak{g}$ and $\xi \in D_X$ then

$$\xi(f) \equiv a_{t+1}\xi(g_{t+1}) + \dots + a_n \xi(g_n) \pmod{\mathfrak{g}}.$$

$(f) \in \int \mathfrak{g}$ if and only if $\xi(f) \equiv 0 \pmod{\mathfrak{g}}$ for any $\xi \in D_X$. By the assumption on $g_{t+1}, \dots, g_n$, we know that $(\xi(g_{t+1}), \dots, \xi(g_n)) = d(g_{t+1}, \dots, g_n)$ is injective at every point on $\Sigma \setminus (X_{\text{sing}} \cup \Sigma_{\text{sing}})$, hence $a_{t+1}, \dots a_n \in \mathfrak{g}$ since (∇) (see [Lo](6.B), or [P2]).

2) Suppose that there exist $\bar{\lambda}_{t+1}, \dots, \bar{\lambda}_n \in \mathcal{O}_\Sigma$ such that $\bar{a} = \bar{\lambda}_{t+1}\bar{g}_{t+1} + \dots + \bar{g}_n \bar{\lambda}_n = 0$ in N. By taking the representatives we have

$$a = \lambda_{t+1} g_{t+1} + \dots + \lambda_n g_n \in (g_1, \dots, g_t) \cap (g_{t+1}, \dots, g_n) + (g_{t+1}, \dots, g_n)^2.$$

Let $a = \lambda_1 g_1 + \dots + \lambda_t g_t + G$, where $G \in (g_{t+1}, \dots, g_n)^2$, then

$$-\lambda_1 g_1 - \dots - \lambda_t g_t + \lambda_{t+1} g_{t+1} + \dots + \lambda_n g_n \in (g_{t+1}, \dots, g_n)^2.$$

Since $g_1, \dots, g_n$ are $\mathcal{O}$-regular, $\lambda_j \in \mathfrak{g}$. Hence $\bar{g}_{t+1}, \dots, \bar{g}_n$ form a free basis of N.$\square$

Remark Note that in the proof the 2), we only used the fact that $g_1, \ldots, g_n$ form an $\mathcal{O}_\Sigma$-regular sequence and the formula in 1).

2.1.4 Let $\mathfrak{g}$ be generated by an $\mathcal{O}_{\mathbb{C}^m}$-regular sequence: $g_1, \ldots, g_n$, and let there exist an integer $0 \leq t \leq n$ and non-zero divisors $\bar{\lambda}_1, \ldots, \bar{\lambda}_t \in \mathcal{O}_\Sigma$ such that $\bar{\lambda}_1\bar{g}_1, \ldots, \bar{\lambda}_t\bar{g}_t$ are zero in M as $\mathcal{O}_\Sigma$-module. Namely

$$\lambda_1 g_1, \ldots, \lambda_t g_t \in \mathfrak{g}^2 + \mathfrak{h}$$

where $\lambda_i g_i$ is the representative of $\bar{\lambda}_i\bar{g}_i$.

Let $h_1 \cdots h_p$ form a minimum generating set of $\mathfrak{h}$. Denote

$$h = (h_1, \cdots, h_p)^T, \quad g = (g_1, \cdots, g_n)^T,$$

$$G = (G_1, \cdots, G_t)^T, \quad \Lambda = \operatorname{diag}\{\lambda_1, \cdots, \lambda_t\},$$

where T means the transposition of the matrix indicated. Let A and $B = (B_1 B_2)$ be the matrices such that

$$\Lambda \left(g_1, \cdots, g_t\right)^T = Ah + G, \quad h = Bg \tag{2.1.5.1}$$

where A is a $t \times p$ matrix, B a $p \times n$ matrix, and B_1 is a $p \times t$ matrix, B_2 is a $p \times (n-t)$ matrix, $G_i \in \mathfrak{g}^2$.

Let $C_1 = AB_1, C_2 = AB_2$, by (2.1.5.1), we have

$$\left(\Lambda - C_1\right)\left(g_1, \cdots, g_t\right)^T - C_2\left(g_{t+1}, \cdots, g_n\right)^T \equiv 0 \quad (\bmod\ \mathfrak{g}^2) \tag{2.1.5.2}$$

Note that $\frac{\mathfrak{g}}{\mathfrak{g}^2}$ is a free $\mathcal{O}_\Sigma$-module. we have

$$\bar{\Lambda} = \bar{C_1}, \quad \bar{C_2} = 0 \text{ in } \mathcal{O}_\Sigma \tag{2.1.5.3}$$

From this we obtain the following proposition similar to the implicit function theorem.

Proposition Let Σ a complete intersection in $\mathbb{C}^m$ defined by $\mathfrak{g}$ as above. There is an open and dense subset Σ^0 of $\Sigma \setminus X_{\text{sing}}$ such that for each $z \in \Sigma^0$,

$$\det C_1 = \det \Lambda(z) = \lambda_1(z) \cdots \lambda_t(z) \neq 0,$$

and consequently

$$\operatorname{rank}(B_1(P)) \geq t \qquad \text{and } t \leq p.$$

Hence from $h_1 = \cdots = h_p = 0$, one can express $g_1, \ldots, g_t$ as functions of $g_{t+1}, \ldots, g_n$ in a neighborhood of every point $P \in \Sigma^0$. $\square$

2.1.5 Freeness of N and the primitive ideal We first quote a theorem of Vasconcelos which is useful for us.

Vasconcelos' Theorem ([Va] 1.1) Let $\mathfrak{a} \supset \mathfrak{b}$ be ideals of a ring R. If $\mathfrak{a}$ is of finite projective dimension, $\mathfrak{a}^2 \subset \mathfrak{b}$, and $\frac{\mathfrak{a}}{\mathfrak{b}} \cong \left(\frac{R}{\mathfrak{a}}\right)^{n-t}$, then there exists an R-regular sequence $a_1, \dots, a_{n-t}$ such that $\mathfrak{a} = (a_1, \dots, a_{n-t}) + \mathfrak{b}$. Moreover, the elements of $\mathfrak{a}$ which are eligible for a minimum generating set of the module $\frac{\mathfrak{a}}{\mathfrak{b}}$ can be taken as the sequence $a_1, \dots, a_{n-t}$.$\square$

Let X and Σ be complete intersections of pure dimensions in $\mathbb{C}^m$ and assume (∇). Let $\mathfrak{g}$ be generated by $\mathcal{O}$-regular sequence $g_1, \dots, g_n$. Then we can choose the generators of $\mathfrak{h}$ such that (with changing of the generators of $\mathfrak{g}$ if necessary):

$$\left\{\begin{array}{lllll} h_1 = & b_{11}g_1 & + \cdots + & b_{1t}g_t & +H_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_p = & b_{p1}g_1 & + \cdots + & b_{pt}g_t & +H_p \end{array}\right. \qquad (2.1.5.1)$$

where t is an integer $0 \leq t \leq n$, and $H_i \in \mathfrak{g}^2$, $b_{ij} \notin \mathfrak{g} \setminus 0$, and for each i, $(b_{i1}, \dots, b_{it}) \neq 0$, that is, for all j, there exists an i such that $b_{ij} \notin \mathfrak{g}$ (otherwise one could lower t). Denote $B := (b_{ij})$.

Lemma Under the assumptions above, we have

1) $t \geq p$;

2) If Σ is irreducible, then there exists at least one non-zero maximal minor of B.

Proof Since X is a complete intersection, the singular locus of X can be defined by the ideal generated by h_i's and the $p \times p$ minors of the Jacobian matrix $J(h)$ of $h := (h_1, \dots, h_p)$ (see [Lo](1.9)). Since each of these minors, say $\Delta_{j_1, \dots, j_p}$, is the determinant of $BG_{j_1, \dots, j_p}$ modulo $\mathfrak{g}$, where $G_{j_1, \dots, j_p}$ is the $t \times p$ submatrix of $J(g)$, consisting of the $0 \leq j_1 < \cdots j_p \leq m$ columns of $J(g)$, the Jacobian matrix of $g := (g_1, \dots, g_n)$. Suppose $t < p$, then on Σ, $\det(BG_{j_1, \dots, j_p}) = 0$. This is in contradiction to the assumption (∇). This proves 1).

2) Suppose that all the $p \times p$ minors of B are zero. Then the rank of B (as a matrix with entries in the quotient field Q of $\mathcal{O}_\Sigma$) will be less than p. Then there are $\lambda_1, \dots, \lambda_p \in Q$ which are not all zero, such that

$$\lambda_1 b_1 + \cdots + \lambda_p b_p = 0 \in Q^t,$$

where b_i is the i-th row vector of B. Let $\lambda_0 \in \mathcal{O}_\Sigma$ be a non-zero element such that $\lambda_0 \lambda_i \in \mathcal{O}_\Sigma$, then

$$\lambda_0 \lambda_1 b_1 + \cdots + \lambda_0 \lambda_p b_p = 0 \in \mathcal{O}_\Sigma^t.$$

Hence

$$\lambda_0\lambda_1 h_1 + \cdots + \lambda_0\lambda_p h_p \equiv 0 \quad (\bmod \mathfrak{g}^2).$$

From this we have $X_{\text{sing}} \supset \Sigma$, a contradiction.$\square$

Proposition Let $\Sigma \subset X$ be complete intersections satisfying (∇). If in (2.1.5.1) we have $t = p$ and $b := \det(b_{ij})$ is a non-zero divisor in $\mathcal{O}_\Sigma$, then

1) $\bar{g}_1, \ldots, \bar{g}_p$ generate $T(M)$ over $\mathcal{O}_\Sigma$;

2) $\bar{g}_{p+1}, \ldots, \bar{g}_n$ generate N freely over $\mathcal{O}_\Sigma$, which implies that (2.1.3.1) splits, so $M = T(M) \oplus N$, and $\text{rank}(M) = \text{rank}(N) = \dim X - \dim \Sigma = n - p$;

3) The sequence $\bar{g}_{p+1}, \ldots, \bar{g}_n$ is $\mathcal{O}_X$- regular, if $\mathfrak{g}$ is of finite projective dimension;

3)* For each $z \in X_{\text{reg}} \cap \Sigma$, the sequence $(\bar{g}_{p+1})_z, \ldots, (\bar{g}_n)_z$ of the germs at z is $\mathcal{O}_{X,z}$-regular;

4) $\int \mathfrak{g} = (g_1, \ldots, g_p) + (g_{p+1}, \ldots, g_n)^2$.

5) When Σ is one dimensional, there is a length formula

$$\varrho(\Sigma X) := l_{\mathcal{O}_\Sigma}(T(M)) = l_{\mathcal{O}_\Sigma}\left(\frac{\mathcal{O}}{(b) + \mathfrak{g}}\right) \tag{2.1.5.2}$$

We call $\varrho(\Sigma X)$ the *torsion number* of $\Sigma \subset X$. When Σ and X are clear from the context, we write ϱ for $\varrho(\Sigma X)$.

Proof Since $t = p$ and b is a non-zero divisor, one can see that $\bar{g}_1, \ldots, \bar{g}_p \in T(M)$ by multiplying B^* to the both sides of (2.1.5.1), where B^* is the adjoint matrix of B.

Since $\bar{g}_1, \ldots, \bar{g}_n$ generate M over $\mathcal{O}_\Sigma$ and

$$0 \longrightarrow T(M) \xrightarrow{i} M \xrightarrow{\pi} N \longrightarrow 0 \tag{2.1.5.3}$$

is eaxct, $\pi(\bar{g}_{p+1}), \ldots, \pi(\bar{g}_n)$ generate N. If there is a relation:

$$\bar{\lambda}_{p+1}\pi(\bar{g}_{p+1}) + \cdots + \bar{\lambda}_n\pi(\bar{g}_n) = 0 \in N,$$

then

$$\bar{\lambda}_{p+1}\bar{g}_{p+1} + \cdots + \bar{\lambda}_n\bar{g}_n \in T(M).$$

This means there is a non-zero divisor $\bar{\lambda} \in \mathcal{O}_\Sigma$ such that

$$\bar{\lambda}(\bar{\lambda}_{p+1}\bar{g}_{p+1} + \cdots + \bar{\lambda}_n\bar{g}_n) = 0 \in M.$$

By taking representatives, this simply means

$$\lambda\lambda_{p+1}g_{p+1} + \cdots + \lambda\lambda_n g_n \in \mathfrak{g}^2 + \mathfrak{h}.$$

Hence there are $\mu_1, \dots \mu_p \in \mathcal{O}$ such that

$$\mu_1 h_1 + \cdots + \mu_p h_p + \lambda\lambda_{p+1} g_{p+1} + \cdots + \lambda\lambda_n g_n \in \mathfrak{g}^2.$$

By (2.1.5.1), this becomes

$$\mu'_1 g_1 + \cdots + \mu'_p g_p + \lambda\lambda_{p+1} g_{p+1} + \cdots + \lambda\lambda_n g_n \in \mathfrak{g}^2,$$

where

$$\begin{pmatrix} \mu'_1 & \cdots \mu'_p \end{pmatrix} = \begin{pmatrix} \mu_1 & \cdots \mu_p \end{pmatrix} B.$$

Since $g_1, \dots, g_n$ form an $\mathcal{O}$-regular sequence, we have $\bar{\lambda}\bar{\lambda}_j = 0$ in $\mathcal{O}_\Sigma$, note that $\bar{\lambda}$ is a non-zero divisor, hence $\bar{\lambda}_j = 0$ in $\mathcal{O}_\Sigma$. This proves 1) and 2).

By using Vasconcelos' Theorem, we have 3).

For each $z \in X_{\mathrm{reg}} \cap \Sigma$,

$$N_z = \frac{\mathfrak{g}_z}{\left(\int \mathfrak{g}\right)_z}$$

is also free with $(\bar{g}_{p+1})_z, \dots, (\bar{g}_n)_z$ as basis. By [P1], $\left(\int \mathfrak{g}\right)_z = \mathfrak{g}_z^2$ since Σ is a reduced complete intersection in z. Since X is regular at z, $\mathcal{O}_{X,z}$ is regular, by Vasconcelos' Theorem, $(\bar{g}_{p+1})_z, \dots, (\bar{g}_n)_z$ is $\mathcal{O}_{X,z}$-regular;

A similar calculation to that in the proof of proposition 2.1.3 will prove the formula for the primitive ideal.

For the length formula, note that

$$T(M) = \frac{\int \mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}} \cong \frac{(g_1, \dots, g_p)}{(g_1, \dots, g_p)^2 + (g_1, \dots, g_p)(g_{p+1}, \dots, g_n) + (\bar{h}_1, \dots, \bar{h}_p)},$$

where $\bar{h}_i = \sum_{j=1}^{p} \bar{b}_{ij} g_j$ is the image of h_i in $\mathcal{O}_\Sigma$ under the canonical mapping.

It is easy to see that

$$M_1 := \frac{(g_1, \dots, g_p)}{(g_1, \dots, g_p)^2 + (g_1, \dots, g_p)(g_{p+1}, \dots, g_n)}$$

is a free $\mathcal{O}_\Sigma$-module. Since b is a non-zero divisor, the following sequence is exact

$$0 \longrightarrow M_1 \xrightarrow{\phi_B} M_1 \longrightarrow T(M) \longrightarrow 0 \qquad (2.1.5.4)$$

where $\phi_B(\bar{g}_i) := \sum_{j=1}^{p} \bar{b}_{ij}\bar{g}_j$. By [F] A.2.6, we have the length formula of $T(M)$.
$\square$

The following example shows that it is not necessary for $T(M)$ to be generated by $\bar{g}_i$ when $t > p$.

Example Let Σ be defined by $\mathfrak{g} = (g_1, g_2)$ with $g_1 = xy, g_2 = z$ and X be defined by $\mathfrak{h} = (h)$ with $h = x^2y + yz + z^2 = xg_1 + yg_2 + g_2^2$. Then $\mathcal{O}_\Sigma \cong \mathbb{C}\{x, y\}/(xy)$, $\int \mathfrak{g} = (x^2y, yz, z^2)$ and $\mathfrak{g}^2 + \mathfrak{h} = (x^2y^2, xyz, z^2, x^2y + yz)$. So $T(M) = \frac{\int \mathfrak{g}}{\mathfrak{g}^2+\mathfrak{h}} \cong \mathbb{C}x^2y$. And N is not a free $\mathcal{O}_\Sigma$-module.

2.1.6 Lines on ***icis*** In this subsection, we consider a special pair Σ and X. Let Σ be a line in $\mathbb{C}^{n+1}$. Define ${}_\Sigma\mathcal{K} := \mathcal{R}_\Sigma \rtimes \mathcal{C}$, the semiproduct of $\mathcal{R}_\Sigma$ with the contact group $\mathcal{C}$ (see Mather[M1]). This group has an action on the space $\mathfrak{m}\mathfrak{g}\mathcal{O}^p$ consisting of mapping germs $h : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}^p, 0), \quad h_j \in \mathfrak{m}\mathfrak{g}$. For $h = (h_1, \dots, h_p) \in \mathfrak{m}\mathfrak{g}\mathcal{O}^p$, we define an analytic space $X = \mathcal{V}(\mathfrak{h})$, where $\mathfrak{h}$ is the ideal generated by $h_1, \dots, h_p$. The image of the morphism:

$$\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p$$

is denoted by $th(h)$, where dh is the differential of h. Define

$$\lambda := \lambda(\Sigma X) = \dim_{\mathbb{C}} \frac{\mathcal{O}^p}{th(h) + \mathfrak{g}\mathcal{O}^p}.$$

Remark that λ is ${}_\Sigma\mathcal{K}$-invariant. The case of $p = 1$ has been considered in §1.4.

Choose Σ as the x-axis. Then Σ can be defined by $\mathfrak{g} = (y_1, \dots, y_n)$.

Theorem Let Σ be a line on an *icis* X defined by $\mathfrak{g}$ and $\mathfrak{h}$ as above. Then h is ${}_\Sigma\mathcal{K}$-equivalent to a mapping germ with components

$$\left\{ \begin{array}{llllll} \tilde{h}_1 = & b_{11}y_1 & & & +H_1 \\ \tilde{h}_2 = & & b_{22}y_2 & & +H_2 \\ \cdots & & \cdots & & \cdots \\ \tilde{h}_p = & & & b_{pp}y_p & +H_p \end{array} \right. \tag{2.1.6.1}$$

where $H_1, \dots, H_p \in \mathfrak{g}^2$ and $b_{ii} \notin \mathfrak{g}$. Moreover

$$\varrho(\Sigma X) = \lambda(\Sigma X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_\Sigma}{(\bar{b})} = \sum_{k=1}^{p} \lambda_k.$$

where $\bar{b}$ is the image of $b := b_{11} \cdots b_{pp}$ in $\mathcal{O}_\Sigma$, and λ_k is the order of $\bar{b}_{kk}$ in $\mathcal{O}_\Sigma$.

Proof Since $\Sigma \subset X$, $\mathfrak{h} \subset \mathfrak{g}$. Then for a given generator set $\{h_1, \dots, h_p\}$ of $\mathfrak{h}$, we have

$$h_k \equiv \sum \bar{b}_{kj} y_j \quad (\bmod \mathfrak{g}^2), \quad k = 1, \dots, p.$$

where $\bar{b}_{kj} \in \mathcal{O}_\Sigma$, and for fixed k, $\bar{b}_{kj}$'s are not all zero since X is complete intersection and $X_{\text{sing}} = \{0\} \subsetneqq \Sigma$ (see the proof of the lemma in §2.1.5).

Since $\mathcal{O}_\Sigma$ is a principal ideal domain, by changing the indices, we can assume $\bar{b}_{11} \mid \bar{b}_{kj}$. Let

$$y_1' = y_1 + \sum_{j=2}^{n} \frac{\bar{b}_{1j}}{\bar{b}_{11}} y_j.$$

Then

$$h_1 \equiv \bar{b}_{11} y_1' \pmod{\mathfrak{g}^2}.$$

Let

$$h_k' = h_k - \frac{\bar{b}_{k1}}{\bar{b}_{11}} h_1, \quad k = 2, \ldots, p.$$

Repeat the above argument will prove the first part of the theorem.

Consider the exact sequence

$$\mathcal{O}^{n+1} \xrightarrow{dh^*} \mathcal{O}^p \longrightarrow \operatorname{coker}(dh^*) \longrightarrow 0.$$

By tensoring with $\mathcal{O}_\Sigma$, we have exact sequence

$$\mathcal{O}_\Sigma^{n+1} \xrightarrow{d\bar{h}^*} \mathcal{O}_\Sigma^p \longrightarrow \operatorname{coker}(d\bar{h}^*) \longrightarrow 0.$$

However by the expression of h_k's above, this is just

$$\mathcal{O}_\Sigma^p \xrightarrow{d\bar{h}^*} \mathcal{O}_\Sigma^p \longrightarrow \frac{\mathcal{O}^p}{th(h) + \mathfrak{g}\mathcal{O}^p} \longrightarrow 0.$$

Since $\bar{b} \neq 0$, by [F] A.2.6, we have the formula for λ.□

Corollary 1 Let X be an *icis* of codimension p in $\mathbb{C}^{n+1}$ and Σ a line in X defined by $\mathfrak{g}$. Then we can choose the coordinates of $\mathbb{C}^{n+1}$ such that $\mathfrak{g} = (y_1, \ldots, y_n)$, $\bar{y}_1, \ldots, \bar{y}_p \in T(M)$ and $\bar{y}_{p+1}, \ldots, \bar{y}_n$ generate N which is free of rank $n-p$, and

$$\int \mathfrak{g} = (y_1, \ldots, y_p) + (y_{p+1}, \ldots, y_n)^2. \qquad \square$$

Example Consider the situation in the example in §2.1.5. Denote $\mathfrak{g}_1 = (x, z)$ and $\mathfrak{g}_2 = (y, z)$. They define $\Sigma_1 = y$-axis and $\Sigma_2 = x$-axis respectively. We have $\mathfrak{g} = \mathfrak{g}_1 \cap \mathfrak{g}_2$ and $\Sigma = \Sigma_1 \cup \Sigma_2 \subset X$. Since $h = (y+z)z + yx^2$, by corollary 1, $\int_X \mathfrak{g}_1 = (x^2, z)$. Since $h = (x^2+z)y + z^2$, by corollary 1, $\int_X \mathfrak{g}_2 = (y, z^2)$. These results tell us that $\int_X \mathfrak{g} = \int_X \mathfrak{g}_1 \cap \int_X \mathfrak{g}_2$.

Corollary 2 (This is a generalization of Pellikaan's example in §1.1.7) Let X be an *icis* of codimension p in $\mathbb{C}^{n+1}$ and Σ a line in X, defined by

$\mathfrak{g} = (y_1, \ldots, y_n)$. Then the Second Exact Sequence (1.1.7.1) is exact on the left also (cf. [H]II 8.17):

$$0 \longrightarrow M \xrightarrow{\delta} \Omega^1_X \otimes \mathcal{O}_\Sigma \longrightarrow \Omega^1_\Sigma \longrightarrow 0 \tag{2.1.6.2}$$

Furthermore it is splitting and

$$T(\Omega^1_X \otimes \mathcal{O}_\Sigma) = T(M), \qquad \mathrm{rank}(\Omega^1_X \otimes \mathcal{O}_\Sigma) = n - p - 1 \tag{2.1.6.3}$$

These tell us that the torsion number $\varrho(\Sigma X)$ is independent of the choice of the generator sets of $\mathfrak{g}$ and $\mathfrak{h}$.

Proof As in the example in §1.1.7, we have the following presentation:

$$\mathcal{O}^p_X \xrightarrow{dh} \mathcal{O}^{n+1}_X \longrightarrow \Omega^1_X \longrightarrow 0.$$

Tensoring with $\mathcal{O}_\Sigma$, we have the exact sequence

$$\mathcal{O}^p_\Sigma \xrightarrow{\overline{dh}} \mathcal{O}^{n+1}_\Sigma \longrightarrow \Omega^1_X \otimes \mathcal{O}_\Sigma \longrightarrow 0.$$

Remark that the map $\overline{dh}$ is equivalent to a map defined by the matrix $(\bar{b}_{ij})$. Hence by (2.1.5.4)

$$\Omega^1_X \otimes \mathcal{O}_\Sigma \cong \frac{\mathcal{O}^{n+1}_\Sigma}{\mathrm{im}\overline{dh}} \cong \mathcal{O}^{n-p+1}_\Sigma \oplus \frac{\mathcal{O}^p_\Sigma}{\mathrm{im}(\bar{b}_{ij})} \cong \mathcal{O}_\Sigma \oplus N \oplus T(M) \cong \mathcal{O}_\Sigma \oplus M \tag{2.1.6.4}$$

Then

$$0 \longrightarrow M \longrightarrow \frac{\mathcal{O}^{n+1}_\Sigma}{\mathrm{im}\overline{dh}} \longrightarrow \mathcal{O}_\Sigma \longrightarrow 0$$

is exact. Since Ω^1_Σ is free $\mathcal{O}_\Sigma$-module of rank 1, by [C] Proposition 6 on page 26 and [F] Example A.2.2, we see that (2.1.6.2) is exact. □

2.1.7 Freeness of N and the primitive ideal (continued) Note that in this subsection, we do not assume (2.1.5.1).

Example Let X be defined by $h := x^3 + xy^3 + 2x^2z + 2z^2 = 0$, Σ be defined by $g_1 := x^2 + y^3 = 0, g_2 := z = 0$. Thus $\mathfrak{h} = (h), \mathfrak{g} = (g_1, g_2)$, and $\Sigma \subset X$. It is easy to check that Σ and X, as germs at 0, have isolated singularity at 0. Notice that X is not weighted homogeneous. So it is not easy to find the generator set of D_X. Then we have the same problem for $\int \mathfrak{g}$. An observation shows that if we denote $g'_1 = g_1 + 2xg_2 + g_2^2$, then $h = xg'_1 + (2-x)g_2^2$, where x is a non-zero divisor in $\mathcal{O}_\Sigma$. By the proposition in §2.1.5, we have:

- $T(M)$ is generated by g'_1 over $\mathcal{O}_\Sigma$

- $\int \mathfrak{g} = (g_1', g_2^2) = (x^2 + y^3 + 2xz, z^2)$
- $N = (g_2)/(g_1' g_2, g_2^2)$ is a free $\mathcal{O}_\Sigma$-module.

The method used in this example can be used to prove a more general

Theorem Let $\Sigma \subset X$ be complete intersections satisfying (∇). If N is a free $\mathcal{O}_\Sigma$-module, then

1) there exists an $\mathcal{O}$-regular sequence $g_1, \dots, g_n$, generating $\mathfrak{g}$, such that

- the images in M of $g_1, \dots, g_p$ generate $T(M)$;
- the images in M of $g_{p+1}, \dots, g_n$ form a basis of N;
- rank(M)=rank(N) $= n - p = \dim X - \dim \Sigma$;

2) let the defining ideal $\mathfrak{h}$ of X be generated by an $\mathcal{O}$-regular sequence $h_1, \dots, h_p$, then we still have (2.1.5.1) with $t = p$ and b a non-zero divisor in $\mathcal{O}_\Sigma$;

3) $\int \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2$;

Proof Let the images of $g_{t+1}, \dots, g_n$ generate N over $\mathcal{O}_\Sigma$, where $n - t =$ rank$N$. By Vasconcelos' theoreom, $g_{t+1}, \dots, g_n$ form an $\mathcal{O}$-regular sequence. Let $\mathfrak{h}$ be generated by $\mathcal{O}$-regular sequence: $h_1, \dots, h_p$.

For $z \in X_{\text{reg}} \cap \Sigma$ near the origin, the stalk N_z is again a free module with generators the images of the germs at z : $(g_{t+1})_z, \dots, (g_n)_z$. Since $\mathcal{O}_{X,z}$ is regular, by Vasconcelos' theorem, the images of $(g_{t+1})_z, \dots, (g_n)_z$ in $\mathcal{O}_{X,z}$ form an $\mathcal{O}_{X,z}$-regular sequence. Hence $(h_1)_z, \dots, (h_p)_z, (g_{t+1})_z, \dots, (g_n)_z$ form an $\mathcal{O}_{\mathbb{C}^m,z}$-regular sequence.

However

$$\dim\{(h_1)_z = 0, \dots, (h_p)_z = 0, (g_{t+1})_z = 0, \dots, (g_n)_z = 0\}$$

$$\geq \dim(\Sigma, z) = \dim\{(g_1)_z = 0, \dots, (g_n)_z = 0\}.$$

We have $n - t + p \leq n$. Hence $t \geq p$.

Extend $g_{t+1}, \dots, g_n$ to an $\mathcal{O}$-regular sequence: $g_1, \dots, g_n$, such that they generate $\mathfrak{g}$. Then $\bar{g}_1, \dots, \bar{g}_n$ generate M over $\mathcal{O}_\Sigma$.

We look for the generator set of $T(M)$. Let

$$\pi(\bar{g}_i) = \bar{c}_{it+1}\pi(\bar{g}_{t+1}) + \cdots + \bar{c}_{in}\pi(\bar{g}_n), \quad i = 1, \dots, t.$$

Then

$$\pi(-\bar{g}_i + \bar{c}_{it+1}\bar{g}_{t+1} + \cdots + \bar{c}_{in}\bar{g}_n) = 0 \in N \quad i = 1, \dots, t.$$

Hence by (2.1.5.3)

$$\bar{g}'_i := -\bar{g}_i + \bar{c}_{it+1}\bar{g}_{t+1} + \cdots + \bar{c}_{in}\bar{g}_n \in T(M), \quad i = 1, \ldots, t.$$

Let

$$g'_i := -g_i + c_{it+1}g_{t+1} + \cdots + c_{in}g_n, \; i = 1, \ldots, t,$$

and

$$g'_{t+j} = g_{t+j}, \; j = 1, \ldots, n-t.$$

Then $\mathfrak{g} = (g'_1, \ldots, g'_n)$, with $\bar{g}'_1, \ldots, \bar{g}'_t \in T(M)$. By the proposition in §2.1.4, $t \leq p$. We have proved 1).

Since $\mathfrak{h} \subset \mathfrak{g}$, we have

$$h_i = b_{i1}g_1 + \cdots + b_{ip}g_p + b_{ip+1}g_{p+1} + \cdots b_{in}g_n, \quad i = 1, \ldots, p.$$

For any $\xi \in D_X$, we have

$$b_{ip+1}\xi(g_{p+1}) + \cdots + b_{in}\xi(g_n) \equiv 0 \quad (\bmod \mathfrak{g}), i = 1, \ldots, p.$$

Using the same argument as in the proof of the proposition 1) in §3.1.3, we know $b_{ip+1}, \ldots, b_{in} \in \mathfrak{g}$ for $i = 1, \ldots, p$. This proves 2).

The formula in 3) is familiar to the reader now. □

2.1.8 Conclusion Let $\Sigma \subset X$ be complete intersections satisfying (∇). The $\mathcal{O}_\Sigma$-module N is free if and only if there exists an $\mathcal{O}$-regular sequence $g_1, \ldots, g_n$, generating $\mathfrak{g}$, such that

$$\int \mathfrak{g} = (g_1, \ldots, g_t) + (g_{t+1}, \ldots, g_n)^2.$$

Proof By the proposition in §2.1.3 2) (see the remark there) and the theorem in §2.1.7.□

2.1.9 Remark If $X_{\text{reg}} = X \setminus \{0\}$, N is free and the images of $g_{p+1}, \ldots, g_n$ form a basis of N. By the proof in §2.1.7, the images of them in $\mathcal{O}_{X,z}$ form an $\mathcal{O}_{X,z}$-regular sequence for $z \in X_{\text{reg}}$. Note that, in general (see example 2 in §2.1.10)

$$\Sigma' := X \cap \mathcal{V}(g_{p+1}, \ldots, g_n) \supsetneqq \Sigma.$$

However, outside the origin, Σ is defined locally by $g_{p+1}, \ldots, g_n$, by this we mean: For any point $P \in \Sigma \setminus \{0\}$, there is an open neighborhood U_P of P in X such that $U_P \subset X_{\text{reg}}$ and $\Sigma \cap U_P = U_P \cap \Sigma'$.

2.1.10 A note on finite projective dimension If N is free and the images of $g_{p+1}, \ldots, g_n$ form a basis of N, then the images of $g_{p+1}, \ldots g_n$

form an $\mathcal{O}_X$-regular sequence if $\bar{\mathfrak{g}}$ is of finite projective dimension: $\mathrm{pd}\bar{\mathfrak{g}} < \infty$. This implies that $h_1, \dots, h_p, g_{p+1}, \dots, g_n$ form an $\mathcal{O}$-regular sequence. Hence $X \cap \mathcal{V}(g_{p+1}, \dots, g_n)$ is a complete intersection.

Practically, given a complete intersection Σ defined by an ideal $\mathfrak{g}$ generated by $\mathcal{O}$-regular sequence $g_1, \dots, g_n$, we have an abundance of complete intersections such that $\mathrm{pd}\bar{\mathfrak{g}} < \infty$. Here is one of the trivial examples. Take X to be defined by $\mathfrak{h} = (g_1, \dots, g_p)$, then $\bar{\mathfrak{g}}$, as an ideal of $\mathcal{O}_X$, is of finite projective dimension. One can easily find some not so trivial examples. For example, if one has $p = t$ in (2.1.5.1) and b a unit, then $\bar{\mathfrak{g}}$ is of finite projective dimension.

However, $\mathrm{pd}\bar{\mathfrak{g}} < \infty$ is a too strong condition for our purpose. $\Sigma \cap U_P = U_P \cap \Sigma'$.

Example 1 Let Σ be the x-axis in $\mathbb{C}^3$ defined by $\mathfrak{g} = (y, z)$, X defined by $h = xy + z^2$. Obviously, the image of z in N generates N freely over $\mathcal{O}_\Sigma$. And the image of z in $\mathcal{O}_X$ is a non-zero divisor.

However, $\bar{\mathfrak{g}}$, as an ideal of $\mathcal{O}_X$, does not have finite projective dimension. Indeed, it is obvious that the Krull dimension of $\mathcal{O}_X$ is 2: $\dim\mathcal{O}_X = 2$. As an $\mathcal{O}_X$-module, $\mathrm{depth}\bar{\mathfrak{g}} = 2$ with $\bar{z}$, $\overline{x+y}$ as a $\bar{\mathfrak{g}}$-regular sequence (this involves some calculations).

Suppose $\mathrm{pd}\bar{\mathfrak{g}} < \infty$, then $\mathrm{pd}\bar{\mathfrak{g}} = \mathrm{height}\bar{\mathfrak{g}} = 1$. Then Auslander-Buchsbaum formula (cf. [BH] page 17) would gives

$$3 = \mathrm{depth}\bar{\mathfrak{g}} + \mathrm{pd}\bar{\mathfrak{g}} = \dim\mathcal{O}_X = 2. \qquad \square$$

Statement Let Σ be a line on a hypersurface X with isolated singularity. Then we can always choose the coordinate system such that Σ is defined by $\mathfrak{g} = (y_1, \dots, y_n)$ and X is defined by $h = ay_1 + \sum_{i,j=2}^{n} a_{ij}y_iy_j = 0$. Then the $\mathcal{O}_\Sigma$-module N is free with the images of $y_2, \dots y_n$ as a basis. Moreover, the images of $y_2, \dots y_n$ in $\mathcal{O}_X$ form an $\mathcal{O}_X$-regular sequence. $\square$

We omit the easy proof of this statement. It indicates that sometimes one can remove the condition of finite projective dimension from Vasconcelos' theorem. Nevertheless, this statement is not true if X is not a hypersurface.

Example 2 Let Σ be the x-axis in $\mathbb{C}^4$ defined by $\mathfrak{g} = (y, z, w)$ and X defined by $h_1 = xy + z^2 + w^2, h_2 = y^2 + xz - w^2$. Obviously, X is an *icis* and N is generated freely by the image of w.

$$\Sigma' := X \cap \mathcal{V}(w) = X \cap \{w = 0\} = \Sigma \cup \{w = 0, x^2 - yz = 0\},$$

which is not a complete intersection in $\mathbb{C}^4$. Hence the image of w in $\mathcal{O}_X$ is not regular. And the projective dimension of $\bar{\mathfrak{g}}$ in not finite.

2.2 Deformations of Non-Isolated Singularities

Our interest in this section is to find a useful deformation of a function f with non-isolated singular locus Σ on a singular space X for studying the topology of the Milnor fibre of f. When X is smooth, this kind of deformations has been first created by Siersma [S3] for functions with singular locus an isolated line. Pellikaan[P2 or 3] has generalized this to functions with singular locus an *icis* . In [deJ-vanS], De Jong and Van Straten have developed a deformation theory for non-isolated singularities. Although their main interest there is the deformations of hypersurfaces with non-isolated singularities, but the general theory, especially the parts A and B in §1 are valid in more general cases. We will go straightforward to find useful deformations, and leave the investigation of the general deformation theory for an other moment. Once more, in finding good deformations, we find that freeness of the torsion free module $M/T(M)$ is essential for us.

2.2.1 The relative Tjurina number Dualizing the complex **LS** in §2.1.1 gives a complex

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_\Sigma}(\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma, \mathcal{O}_\Sigma) \xrightarrow{\partial_1{}^*} \mathcal{O}^n_\Sigma \xrightarrow{\partial_2{}^*} \mathrm{Hom}_{\mathcal{O}_\Sigma}(H_1(\bar{\mathfrak{g}}, \mathcal{O}_\Sigma), \mathcal{O}_\Sigma) \longrightarrow 0 \tag{2.2.1.1}$$

Since

$$\mathrm{Hom}_{\mathcal{O}_\Sigma}(\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma, \mathcal{O}_\Sigma) \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_\Sigma) = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma).$$

There is a morphism

$$\phi : \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma) \tag{2.2.1.2}$$

defined by $\phi(\xi \otimes a)(\omega) = \xi(\omega) \otimes a$. If X is smooth, this is an isomorphism since Ω^1_X is free. In general, one needs the flatness of $\mathcal{O}_\Sigma$ as an $\mathcal{O}_X$-module. However this is not the case in general.

Example (This example is due to Pellikaan) Take $X : h = xy + z^2 = 0$. And Σ is defined by $\mathfrak{g} = (y, z)$. Then there is a presentation:

$$\mathcal{O}_X \xrightarrow{dh} \mathcal{O}^3_X \longrightarrow \Omega^1_X \longrightarrow 0,$$

where $dh = \begin{pmatrix} y & x & 2z \end{pmatrix}^T$ as a matrix. Dualizing with $\mathrm{Hom}(-, \mathcal{O}_\Sigma)$ gives left exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_\Sigma) \longrightarrow \mathcal{O}^3_\Sigma \xrightarrow{dh^*} \mathcal{O}_\Sigma,$$

where we have $dh^* = \begin{pmatrix} 0 & x & 0 \end{pmatrix}$ as a matrix. Hence $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_\Sigma) \cong \mathcal{O}^2_\Sigma$ is a free $\mathcal{O}_\Sigma$-module of rank 2 with $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ as a basis.

However $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) = D_X$ is generated over $\mathcal{O}_X$ by

$$\xi_0 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},\ \xi_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y},\ \xi_2 = 2z\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ \xi_3 = 2z\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}.$$

Then

$$\phi(\xi_0) = \phi(\xi_1) = x\frac{\partial}{\partial x}, \quad \phi(\xi_2) = 0, \quad \phi(\xi_3) = -x\frac{\partial}{\partial z},$$

so ϕ is not surjective.□

Nevertheless, let $\partial^0 := \partial_1^* \circ \phi$, we have a complex of $\mathcal{O}_\Sigma$-modules and morphisms:

$$0 \longrightarrow D_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma \xrightarrow{\partial^0} \mathrm{Hom}_{\mathcal{O}_\Sigma}(\mathcal{O}^n_\Sigma, \mathcal{O}_\Sigma) \xrightarrow{\partial^1} \mathrm{Hom}_{\mathcal{O}_\Sigma}(H_1(\bar{\mathfrak{g}}, \mathcal{O}_\Sigma), \mathcal{O}_\Sigma) \longrightarrow 0 \qquad (2.2.1.1)'$$

where $D_X = \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$, and

$$\partial^0(\xi) = \sum \xi(g_i)e_i^*, \quad \partial^1(\alpha = \sum a_i e_i^*) : r = \sum r_i e_i \mapsto \sum a_i r_i.$$

The cohomology groups of $(2.2.1.1)'$ are denoted by $T^0_{\Sigma X}, T^1_{\Sigma X}$ and $T^2_{\Sigma X}$. It is easy to see that

- $T^0_{\Sigma X} = T\mathcal{R}^e_{\Sigma X} \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma = \{\xi \mid \xi(\mathfrak{h}) = \mathfrak{h} \text{ and } \xi(\mathfrak{g}) = \mathfrak{g}\} \otimes \mathcal{O}_\Sigma$, which is the $\mathcal{O}_\Sigma$-module of the vector fields of X on Σ, the infinitesimal automorphisms of Σ in X. We denote this module by $\Theta_{\Sigma X}$;
- $T^2_{\Sigma X} = 0$ if Y is strongly Cohen-Macaulay subscheme of X (see §2.1.1).

It is easy to see that the kernel of ∂^1 is the *normal module*

$$M^* := \mathrm{Hom}(M, \mathcal{O}_\Sigma).$$

Then we have an exact sequence

$$0 \longrightarrow \Theta_{\Sigma X} \longrightarrow D_X \otimes_{\mathcal{O}_X} \mathcal{O}_\Sigma \xrightarrow{\partial^0} M^* \longrightarrow T^1_{\Sigma X} \longrightarrow 0 \qquad (2.2.1.3)$$

When X is smooth, $T^1_{\Sigma X}$ is the *infinitesimal deformations* of Σ, and its dimension $\dim_{\mathbb{C}} T^1_{\Sigma X}$ is called the *Tjurina number.* For a singular space X, we call this dimension the *relative Tjurina number,* denoted by

$$\tau(\Sigma X) := \dim_{\mathbb{C}} T^1_{\Sigma X}$$

if it is finite.

Proposition Let $\Sigma \subset X$ be a line in $(\mathbb{C}^{n+1}, O)$ defined by $\mathfrak{g} = (y_1, \dots, y_n)$ and $(X, O) \subset (\mathbb{C}^{n+1}, O)$ an isolated complete intersection singularity defined by ideal $\mathfrak{h}$ generated by $h_1, \dots, h_p$ given by (2.1.5.1). If $h_1, \dots, h_p$ are weighted homogeneous with respect to the coordinates $(x, y_1, \dots, y_n)$ of $(\mathbb{C}^{n+1}, 0)$, then

$$\tau(\Sigma X) = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_\Sigma}{(\bar{b}^{n-p})} \right),$$

where $b = \det(b_{ij})$.

Proof From the corollary 1 in §2.1.6, $M \cong T(M) \oplus \mathcal{O}_\Sigma^{n-p}$, $M^* \cong \mathcal{O}_\Sigma^{n-p}$. It is well known that the trivial derivations of X are given by the $(p+1) \times (p+1)$-minors of the matrix

$$J(-, h) = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y_1} & \cdots & \frac{\partial}{\partial y_n} \\ \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_p}{\partial x} & \frac{\partial h_p}{\partial y_1} & \cdots & \frac{\partial h_p}{\partial y_n} \end{pmatrix}.$$

Let $\xi_{j_0, j_1, \dots j_p}$ be the derivation given by the minor consisting of the $0 \le j_0 < j_1 < \cdots < j_p \le n$-th columns of $J(-, h)$. Let $\xi^j = \xi_{1, \dots, p, p+j} \quad (j = 1, \dots, n-p)$. Then ξ^j takes the following form

$$\xi^j = \xi_1^j \frac{\partial}{\partial y_1} + \cdots + \xi_p^j \frac{\partial}{\partial y_p} + (b + b') \frac{\partial}{\partial y_{p+j}},$$

where $b' \in \mathfrak{g}$. Let $D'_X \subset D_X$ be the submodule of D_X generated by $\xi^1, \dots, \xi^{n-p}$ freely.

Hence

$$D_X \otimes \mathcal{O}_\Sigma = \{\xi_E\}\mathcal{O}_\Sigma + \mathcal{O}_\Sigma\{\xi_{j_0, j_1, \dots j_p}\}_{(j_0, j_1, \dots j_p) \ne (1, \dots, p, p+j)} + D'_X \otimes \mathcal{O}_\Sigma.$$

And ∂^0 is equivalent to

$$D'_X \otimes \mathcal{O}_\Sigma = \xi^1 \mathcal{O}_\Sigma + \cdots + \xi^{n-p} \mathcal{O}_\Sigma \cong \mathcal{O}_\Sigma^{n-p} \xrightarrow{bE} \mathcal{O}_\Sigma^{n-p} \cong M^*,$$

where E is the $(n-p) \times (n-p)$ unit matrix. Since $b \ne 0$, by [F] A.2.6

$$\tau(\Sigma X) = \dim(\operatorname{coker}(\partial^0)) = \dim(\operatorname{coker}(bE)) = \dim \left(\frac{\mathcal{O}_\Sigma}{\det(bE)} \right). \qquad \square$$

Remark If Σ is a line, X is a weighted homogeneous *icis*, then $\varrho(\Sigma X) \le p\tau(\Sigma X)$. Especially, when X is a weighted homogeneous hypersurface with isolated singularity and $n = 2$, then $\varrho(\Sigma X) = \tau(\Sigma X)$.

Example Take Σ to be the union of x and y-axis in $\mathbb{C}^3$ (not smooth) defined by $\mathfrak{g} = (xy, z)$, and X_k the surface singularities of type A_k defined by $h_k = xy + z^{k+1} = 0$, we have

$$\tau(\Sigma X_k) = 1, \quad (k = 1, 2, \dots ,).$$

Since $\int \mathfrak{g} = (xy, z^2) = \mathfrak{g}^2 + (h_k)$, $\varrho(\Sigma X_k) = 0, \quad (k = 1, 2, \dots ,)$.

2.2.2 The transversal Hessian Following the way of Pellikaan (see [P3]), we make the following construction. Let $f \in \mathfrak{g}^2$, write $f = \sum h_{ij} g_i g_j$ for $h_{ij} \in \mathcal{O}$ with $h_{ij} = h_{ji}$. Define maps $\tilde{h}_f : M^* \longrightarrow M$ by composing the maps:

$$M^* \xrightarrow{g^*} \mathcal{O}_\Sigma^n \xrightarrow{(h_{ij})} \mathcal{O}_\Sigma^n \xrightarrow{g} M \tag{2.2.2.1}$$

and $h_f : M^* \longrightarrow N$ by composing the maps:

$$M^* \xrightarrow{\tilde{h}_f} M \xrightarrow{\pi} M' \tag{2.2.2.2}$$

where $M^* := \mathrm{Hom}_{\mathcal{O}_\Sigma}(M, \mathcal{O}_\Sigma)$ and $M' := M/T(M)$.

A similar argument to [P3] 1.7 gives:

Lemma The map h_f depends only on f and $\mathfrak{g}$.□

Definition For $f \in \mathfrak{g}^2$, define

$$\tilde{\delta}_f = \dim_{\mathbb{C}}(\mathrm{coker}(\tilde{h}_f)), \qquad \delta_f = \dim_{\mathbb{C}}(\mathrm{coker}(h_f)).$$

2.2.3 Lemma (see [P3] 1.10) Let $\Sigma \subset X$ satisfy (∇), and Σ a local complete intersection in X_{reg}. Let $f \in \mathfrak{g}^2$. Then $\delta_f = 0$ if and only if N is free and one can choose the generators $g_1, \dots , g_n$ of $\mathfrak{g}$ such that $\bar{g}_{t+1}, \dots , \bar{g}_n$ form a free basis of the $\mathcal{O}_\Sigma$-module N, and

$$f = f' + \sum_{i,j=t+1}^{n} h_{ij} g_i g_j$$

with $f' \in (g_1, \dots , g_t)\mathfrak{g}$ and $\det(h_{ij})$ is a unit in $\mathcal{O}$.

Proof By the lemma in §2.1.2 we have $M^* \cong N^*$. Since $\delta_f = 0$, exact is

$$N^* \xrightarrow{h_f} N \longrightarrow 0,$$

which factorizes as in diagram 2.2.1, where θ is the canonical map, since h_f is induced by symmetric matrix. Hence h_f is an isomorphism and makes the exact sequence (2.1.3.1) split (see diagram 2.2.2). So $M = N \oplus T(M)$.

$$\begin{tikzcd} 0 \arrow[r] & N^* \arrow[r, "h_f^*"] \arrow[dr, "h_f"'] & N^{**} \\ & & N \arrow[u, "\theta"'] \end{tikzcd}$$

Diagram 2.2.1

$$\begin{tikzcd} 0 \arrow[r] & T(M) \arrow[r] & M \arrow[r, "\pi"] & N \arrow[r] & 0 \\ & & \mathcal{O}_\Sigma^n \arrow[u, "g"] & N^* \arrow[u, "h_f"'] \arrow[d, "\cong"] & \\ & & \mathcal{O}_\Sigma^n \arrow[u, "(h_{ij})"] & M^* \arrow[l, "g^*"] & \end{tikzcd}$$

Diagram 2.2.2

Let $\mathcal{O}$-regular sequence $g_1, \dots, g_n$ be the generators of $\mathfrak{g}$ such that the projections of $g_{t+1}, \dots, g_n$ are the minimal generator set of N. Let $f = \sum h_{ij} g_i g_j$. Let $\hat{h} : \mathcal{O}_\Sigma^{n-t} \longrightarrow \mathcal{O}_\Sigma^{n-t}$ be the map induced by the $(n-t) \times (n-t)$- symmetric matrix $(h_{ij})_{t+1 \le i,j \le n}$ and $\hat{g} : \mathcal{O}_\Sigma^{n-t} \longrightarrow N$ be defined by $\hat{g}(\lambda_{t+1}, \dots, \lambda_n) = \lambda_{t+1}\bar{g}_{t+1} + \cdots + \lambda_n \bar{g}_n$.

It is obvious that the composition $\hat{h}_f$ of

$$N^* \xrightarrow{\hat{g}^*} \mathcal{O}_\Sigma^{n-t} \xrightarrow{\hat{h}} \mathcal{O}_\Sigma^{n-t} \xrightarrow{\hat{g}} N$$

coincides with h_f. But h_f is an isomorphism. So N is a direct summand of $\mathcal{O}_\Sigma^{n-t}$. They have the same minimal number of generators, hence $N = \mathcal{O}_\Sigma^{n-t}$.

Hence $(h_{ij})_{t+1 \le i,j \le n}$ is invertible, and $\bar{g}_{t+1}, \dots, \bar{g}_n$ form a basis of the $\mathcal{O}_\Sigma$-module N. $\square$

Proposition (see [P3] 1.12) Let Σ be a local complete intersection in X, and $X_{\mathrm{sing}} = \{0\}$. If $f \in \mathfrak{g}^2$, $j(f) < \infty$, then

1) h_f is injective;

2) The sequence

$$0 \longrightarrow T(M) \longrightarrow \mathrm{coker}(\tilde{h}_f) \longrightarrow \mathrm{coker}(h_f) \longrightarrow 0 \qquad (2.2.3.1)$$

is exact. Consequently

$$\tilde{\delta}_f = \delta_f + \varrho \qquad (2.2.3.2)$$

3) $\boxed{j(f) = c^e(f) + \tau(\Sigma X) + \delta_f.}$

Proof Repeating the argument of [P3] 1.12 and 1.13 with some modification will prove 1) and 3).

Since

$$\mathrm{im}(h_f) = \pi(\mathrm{im}\tilde{h}_f) = \frac{\mathrm{im}\tilde{h}_f + T(M)}{T(M)}, \quad \mathrm{coker}(h_f) = \frac{M}{\mathrm{im}\tilde{h}_f + T(M)},$$

we have exact sequence

$$0 \longrightarrow \frac{\mathrm{im}\tilde{h}_f + T(M)}{\mathrm{im}\tilde{h}_f} \longrightarrow \frac{M}{\mathrm{im}\tilde{h}_f} \longrightarrow \frac{M}{\mathrm{im}\tilde{h}_f + T(M)} \longrightarrow 0.$$

To prove the exactness of (2.2.3.1), it is enough to prove

$$\frac{\mathrm{im}\tilde{h}_f + T(M)}{\mathrm{im}\tilde{h}_f} = \frac{T(M)}{\mathrm{im}\tilde{h}_f \cap T(M)} = T(M)$$

which is equivalent to $\mathrm{im}\tilde{h}_f \cap T(M) = 0$. Let $a \in \mathrm{im}\tilde{h}_f \cap T(M)$ and $a = \tilde{h}_f(b) \in T(M)$, then $h_f(b) = \pi(a) = 0$. We get $b = 0$ since h_f is injective, hence $a = 0$.□

2.2.4 The computation of δ_f In this subsection, we discuss the relationship of $\tilde{\delta}_f$ δ_f and ϱ and how to compute them in practice. Let $M' = M/T(M)$ be the torsion free factor module.

From the exact and commutative diagram 2.2.3 we have

$$\tilde{\delta}_f = \dim_{\mathbb{C}}(\mathrm{coker}\tilde{h}_f) = \dim_{\mathbb{C}}(\mathrm{coker}\tilde{h}'_f) + \varrho \tag{2.2.4.1}$$

If we assume (∇) and Σ to be a complete intersection in $\mathbb{C}^m$, then $T(M) = \frac{\int \mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}}$, $M' = N$ and $\tilde{h}'_f = h_f$. Hence

$$\delta_f = \tilde{\delta}_f - \varrho = e_{\mathcal{O}_\Sigma}(h_f) + \dim_{\mathbb{C}}(\mathrm{ker}h_f) \tag{2.2.4.2}$$

where $e_{\mathcal{O}_\Sigma}(h_f) := \dim_{\mathbb{C}}(\mathrm{coker}h_f) - \dim_{\mathbb{C}}(\mathrm{ker}h_f)$, by definition, which has many good properties (see [F] §A.2).

If N is free, then $N \cong M^*$. Moreover, if the determinant of the matrix of h_f under a free basis of N is not zero in $\mathcal{O}_\Sigma$, then by [F] A.2.6

$$\delta_f = \tilde{\delta}_f - \varrho = e_{\mathcal{O}_\Sigma}(\det(h_f)) + \dim_{\mathbb{C}}(\mathrm{ker}h_f) \tag{2.2.4.3}$$

where we view $\det(h_f)$ as an endomorphism of $\mathcal{O}_\Sigma$ by multiplying with $\det(h_f)$. Especially when $\det(h_f)$ is not a zero divisor, then h_f is injective, and we have

$$\boxed{\delta_f = \tilde{\delta}_f - \varrho = \dim_{\mathbb{C}} \frac{\mathcal{O}_\Sigma}{(\det(h_f))}} \tag{2.2.4.4}$$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[r] \arrow[d] & T(M) \arrow[r] \arrow[d] & \mathrm{ker} q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{ker}\tilde{h}_f \arrow[r] \arrow[d] & M^* \arrow[r, "\tilde{h}_f"] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, "p"] & \mathrm{coker}\tilde{h}_f \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & \mathrm{ker}\tilde{h}'_f \arrow[r] \arrow[d] & M'^* \arrow[r, "\tilde{h}'_f"] \arrow[d] & M' \arrow[r] \arrow[d] & \mathrm{coker}\tilde{h}'_f \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Diagram 2.2.3

If N is free, then $N \cong M^*$. Moreover, if the determinant of the matrix of h_f under a free basis of N is not zero in $\mathcal{O}_\Sigma$, then by [F] A.2.6

$$\delta_f = \tilde{\delta}_f - \varrho = e_{\mathcal{O}_\Sigma}(\det(h_f)) + \dim_{\mathbb{C}}(\mathrm{ker} h_f) \tag{2.2.4.3}$$

where we view $\det(h_f)$ as an endomorphism of $\mathcal{O}_\Sigma$ by multiplying with $\det(h_f)$. Especially when $\det(h_f)$ is not a zero divisor, then h_f is injective, and we have

$$\boxed{\delta_f = \tilde{\delta}_f - \varrho = \dim_{\mathbb{C}} \frac{\mathcal{O}_\Sigma}{(\det(h_f))}} \tag{2.2.4.4}$$

2.2.5 Admissible Deformations Let $\Sigma \subset X$ be a pair of purely dimensional analytic space germs embedded in a neighborhood of $0 \in \mathbb{C}^m$, defined by radical ideals $\mathfrak{g} \supset \mathfrak{h}$ of $\mathcal{O}_{\mathbb{C}^m,0}$ respectively. Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be an analytic function. Let $Y = X \cap f^{-1}(0)$, and Σ_f be the singular locus of f. Obviously, $Y_{\mathrm{sing}} = \Sigma_f$.

Let S be a space, a *deformation of a space Z over S* is a flat morphism $Z_S \xrightarrow{\pi} S$ such that $Z \cong Z_S \times_S \mathrm{Spec}(\mathbb{C})$.

A *deformation* of $\Sigma \hookrightarrow X$ consists of deformations of Σ and X: $\Sigma_S \longrightarrow S$, $X_S \longrightarrow S$ and a morphism $\Sigma_S \hookrightarrow X_S$ such that diagram 2.2.4 is commutative.

Let $\Sigma_S \hookrightarrow X_S$ be a deformation of $\Sigma \hookrightarrow X$, and $f_S : (X_S, 0) \longrightarrow (\mathbb{C}, 0)$ be an analytic function. (f_S, Σ_S, X_S) is called an *admissible deformation* (see [deJ-vanS]) of (f, Σ, X) if $Y_S := X_S \cap f_S^{-1}(0)$ is a deformation of Y, the restriction f_0 of f_S to the fibre X of X_S is f and the diagram 2.2.5 is commutative, where Σ_{f_S} is the singular locus of f_S.

$$\begin{tikzcd} \Sigma_S \arrow[r] \arrow[dr] & X_S \\ & S \arrow[u] \end{tikzcd} \qquad\qquad \begin{tikzcd} \Sigma_S \arrow[r] \arrow[dr] & Y_S \\ & \Sigma_{f_S} \arrow[u] \end{tikzcd}$$

Diagram 2.2.4 Diagram 2.2.5

Let $(S, 0)$ be embedded in $(\mathbb{C}^\sigma, 0)$, $X_S = X \times S \subset \mathbb{C}^m \times \mathbb{C}^\sigma$. Denote $\tilde{\mathcal{O}} := \mathcal{O}_{\mathbb{C}^m \times \mathbb{C}^\sigma, 0}$. Then X_S can be defined by a radical ideal $\tilde{\mathfrak{h}} \subset \tilde{\mathcal{O}}$. Let Σ_S be defined by a radical ideal $\tilde{\mathfrak{g}} \subset \tilde{\mathcal{O}}$ such that $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$. Denote $\tilde{D}_X = D_X \otimes \tilde{\mathcal{O}}$. Then (f_S, Σ_S, X_S) is an admissible deformation of (f, Σ, X) if and only if

$$f_S \in \int \tilde{\mathfrak{g}} := \left\{ g \in \tilde{\mathfrak{g}} \mid \tilde{\xi}(g) \in \tilde{\mathfrak{g}}, \forall \tilde{\xi} \in \tilde{D}_X \right\}.$$

Define

$$F : (\mathbb{C}^m \times \mathbb{C}^\sigma, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^\sigma, 0) \text{ by } F(z, u) := (f_S(z, u), u)$$

Denote $\tilde{f} := f_S$ and $\tilde{J}_X(\tilde{f}) := \left\{ \xi(\tilde{f}) \mid \xi \in \tilde{D}_X \right\}$. If (f_S, Σ_S, X_S) is an admissible deformation of (f, Σ, X), we sometimes call F, f_S , $\tilde{f}$ or $f_u := f_S(-, u)$ to be a deformation of f.

If we consider F as a map on X_S, then the critical locus of F can be defined by

$$C_F := \left\{ (z, u) \in X \times S \mid \xi(\tilde{f})(z, u) = 0, \xi \in \tilde{D}_X \right\} = \mathcal{V}(\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}}).$$

2.2.6 The Fitting ideals of the Jacobian module Since $(\Sigma_S, 0) \subset (C_F, 0)$. Hence $F|_{C_F}$ is not a finite map and we can not expect the discriminant of F to be analytic (see [Te], [BR], [P3] and [DGW]). In fact, the finiteness of $F|_{C_F}$ is a characteristic property of isolated singularity (see *loc.cit.*). There is, however, some well defined subspace of C_F (called residual critical locus) playing the rôle of the critical locus in the isolated case (see [P3]). Let us now describe this subspace. Let

$$\tilde{W} = \frac{\tilde{\mathfrak{g}}}{\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}}},$$

which is a finitely generated $\tilde{\mathcal{O}}$-module. There is a presentation:

$$\tilde{\mathcal{O}}^{q_1} \xrightarrow{\phi} \tilde{\mathcal{O}}^{q_0} \longrightarrow \tilde{W} \longrightarrow 0,$$

where, under the canonical basis of the relative modules, ϕ can be taken as a $q_0 \times q_1$-matrix with entries in $\tilde{\mathcal{O}}$. The $k-$th *Fitting ideal* of $\tilde{W}$, denoted by $\mathcal{F}_k(\tilde{W})$, is by definition the ideal of $\tilde{\mathcal{O}}$ generated by the $(q_0 - k) \times (q_0 - k)$-minors of ϕ, and $\mathcal{F}_k(\tilde{W}) = \tilde{\mathcal{O}}$ in case $k \geq q_0$ and $\mathcal{F}_k(\tilde{W}) = 0$ in case $q_0 > q_1$. Note that Fitting ideals do not depend on the choice of the presentation of the module and is stable under base exchange (see [Lo] for details). Especially, $\mathcal{F}_0(\tilde{W}) \subset \mathrm{Ann}(\tilde{W})$, and both of them define $\mathrm{Supp}(\tilde{W})$.

It is a good exercise to verify that $\mathcal{F}_0(W) = R$ if and only if $W = 0$ for any finitely generated module W over a commutative ring R with 1.

Proposition If $f \in \int_X \mathfrak{g}$ and $j(f) < \infty$, then

$$\dim_{\mathbb{C}} \frac{\mathcal{O}}{J_X(f) + \mathfrak{h} + \mathcal{F}_0(W)} < \infty,$$

where $W := \frac{\mathfrak{g}}{J_X(f) + \mathfrak{h}}$.

Proof Use the Nullstellensatz for coherent sheaves.

2.2.7 Residual critical locus and the Jacobian number We have

$$\mathrm{Supp}(\tilde{W}) = \mathcal{V}((\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}}) : \tilde{\mathfrak{g}}) = \mathcal{V}(\mathcal{F}_0(\tilde{W})) = \mathcal{V}(\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}} + \mathcal{F}_0(\tilde{W})),$$

and since

$$(\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}})^2 \subset ((\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}}) : \tilde{\mathfrak{g}})\tilde{\mathfrak{g}} \subset \tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}},$$

$$C_F = \Sigma_S \cup \mathcal{V}(\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}} + \mathcal{F}_0(\tilde{W})).$$

The analytic subspace of X_S :

$$RC_F := \mathcal{V}(\tilde{J}_X(\tilde{f}) + \tilde{\mathfrak{h}} + \mathcal{F}_0(\tilde{W}))$$

is called the *residual critical locus* of the deformation F of f.

Theorem (see [P2] 2.13) Let $f \in \int_X \mathfrak{g}$ and $j(f) < \infty$.

1) The restriction germs

$$F|_{RC_F} : RC_F \longrightarrow (\mathbb{C} \times \mathbb{C}^\sigma, 0)$$

and the composition p_2 of $F|_{RC_F}$ with $\pi : (\mathbb{C} \times \mathbb{C}^\sigma, 0) \longrightarrow (\mathbb{C}^\sigma, 0)$:

$$p_2 : RC_F \longrightarrow (\mathbb{C}^\sigma, 0)$$

are finite maps;

2) The *residual discriminant* $RD_F := \mathrm{im}(F|_{RC_F})$ of F is an analytic space in $(\mathbb{C}^\sigma, 0)$, defined by $\mathcal{F}_0(F_*(\tilde{W}))$;

3) There exist representatives of all the germs considered such that for all $s \in S$

$$j(f) \geq \sum_{P \in \pi^{-1}(s)} j(f_{s,P}),$$

where $j(f_{s,P})$ is the Jacobian number of f_s at point P. ([1])

Proof 1) and 2) are obvious by using the proposition in §2.2.6, [Gu] Theorem 5(d) and the finite mapping theorem.

For 3), let

$$\bar{W} := \frac{\tilde{W}}{\pi^*(\mathfrak{m}_{\mathbb{C}^\sigma, o})\tilde{W}}$$

which is an $\mathcal{O}_{RC_F}$-finite module with finite length: $l(\bar{W}) = j(f)$.

We consider $\mathcal{O}_{RC_F}$-finite module $\tilde{W}$. Still denote by $\tilde{W}$ the sheaf defined by $\tilde{W}$ on RC_F (in a neighborhood of 0). Since p_2 is finite, $(p_2)_*\tilde{W}$ is a finite $\mathcal{O}_{\mathbb{C}^\sigma}$-module (in fact an $\mathcal{O}_{RD_F}$-finite module). Remark that for any $s \in \mathbb{C}^\sigma$ near 0, we have

$$\left((p_2)_*\tilde{W}\right)_s = \bigoplus_{P \in p_2^{-1}(s)} \tilde{W}_P$$

as $\mathcal{O}_{\mathbb{C}^\sigma}$-module. Then

$$\dim_{\mathbb{C}} \left(\bigoplus_{P \in p_2^{-1}(s)} \bar{W}_P \right) = \sum_{P \in p_2^{-1}(s)} \dim_{\mathbb{C}} \left(\bar{W}_P\right) = \sum_{P \in p_2^{-1}(s)} j(f_{s,P})$$

By taking sufficient small neighborhood of $0 \in \mathbb{C}^\sigma$, we may assume that the minimal generator set of $\tilde{W}_0$ consists of $j(f)$ elements. Since $p_2^{-1}(0) = \{0\}$,

[1] Is it possible to have a equality ?

we have

$$\dim_{\mathbb{C}}\left((p_2)_*\bar{W}\right)_0 = \dim_{\mathbb{C}}\bar{W} = j(f) \geq \sum_{P\in p_2^{-1}(s)} j(f_{s,P}). \quad \square$$

2.2.8 A good deformation Let $X \subset \mathbb{C}^m$ be a reduced pure dimensional analytic space with isolated singularity at 0, defined by an ideal $\mathfrak{h} \subset \mathcal{O}$, Σ a reduced curve in X with isolated singularity at 0, defined by an ideal $\mathfrak{g} = (g_1, \dots, g_n) \subset \mathcal{O}$. Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ and $f \in \mathfrak{g}^2$. We can write $f = \sum_{k,l=1}^{n} h_{kl}g_kg_l$, $h_{kl} = h_{lk}$. Let $u = (u_{kl})_{1\leq k,l\leq n}$ be the coordinates of $\mathbb{C}^{n^2}$. Let $U = \{u \in \mathbb{C}^{n^2} \mid u_{kl} = u_{lk}\}$, and $V = \mathbb{C}^{mn}$ with coordinates $v = (v_{jk})_{1\leq j\leq m, 1\leq k\leq n}$. Let $s = (u, v)$ be the coordinates of $S = U \times V$. Define

$$f_s(z) = f + \sum_{k,l=1}^{n}\left(u_{kl} + \sum_{j=0}^{n} z_jv_{jk}\delta_{kl}\right)g_kg_l,$$

where δ_{kl} is Kronecker's delta. We call f_s a *good deformation* of f used originally by Siersma when X is smooth (see [S3-5] and [P2, 3]).

2.2.9 Theorem Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be an *icis* of pure dimension $n - p + 1$, $(\Sigma, 0) \subset (\mathbb{C}^{n+1}, 0)$ an *icis* of pure dimension 1. If $N := M/T(M)$ is a free $\mathcal{O}_\Sigma$-module and $j(f) < \infty$, then there exists a open dense subset S' of a neighbourhood of 0 in S such that for any $s \in S'$

1) on $X \setminus \Sigma$, f_s has only A_1 type critical points, and

2) on $\Sigma \setminus 0$, f_s has only A_∞ and D_∞ singularities.

Proof 1) Write $X^1 = X \setminus \Sigma$, which is a manifold. Consider a map germ $\Phi : X^1 \times S \longrightarrow T^*X^1$ by $\Phi(z, s) = df_s(z) \in T^*_zX^1$, where T^*X^1 is the cotangent bundle of X^1. For any $P \in X^1$, let Z be a neighborhood of P in X^1 on which one of the minors of $J(h)$, for example, the minor consisting of the first p columns $\Delta_{1,\dots,p} \neq 0$, where $J(h)$ is the Jacobian matrix of the map $h := (h_1, \dots, h_p) : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^p$. By implicit function theorem, we have

$$z_j = \phi_j(z_{p+1}, \dots, z_m), \quad j = 1, \dots, p$$

from $h_1 = 0, \dots, h_p = 0$ which are the generators of the defining ideal $\mathfrak{h}$ of X, where $m = n + 1$.

Denote $\psi_k = \phi_k$ when $1 \leq k \leq p$ and $\psi_k = z_k$ when $p+1 \leq k \leq m$. Replace

$z_1, \ldots, z_p$ in f_s by $\phi_1, \ldots, \phi_p$ respectively. We have

$$\tilde{f}_s(z_{p+1}, \ldots, z_m) = f_s(\psi_1, \ldots, \psi_m) = \sum_{k,l=1}^{n} \left(\tilde{h}_{kl} + \tilde{u}_{kl} + \sum_{j=0}^{m} \psi_j v_{jk} \delta_{kl} \right) \tilde{g}_k \tilde{g}_l.$$

Notice that $z_{p+1}, \ldots, z_m$ are the local coordinates on Z. In this piece of X^1, $\Phi(z,s) = df_s(z)$ can be expressed , in the local coordinates, by

$$\Phi(z,s) = \left(\frac{\partial \tilde{f}_s}{\partial z_{p+1}}, \ldots, \frac{\partial \tilde{f}_s}{\partial z_m} \right).$$

Then

$$\frac{\partial^2 \tilde{f}}{\partial u_{kk} \partial z_i} = 2\tilde{g}_k \frac{\partial \tilde{g}_k}{\partial z_i}, \quad \frac{\partial^2 \tilde{f}}{\partial v_{jk} \partial z_i} = 2\psi_j \tilde{g}_k \frac{\partial \tilde{g}_k}{\partial z_i} + \frac{\partial \psi_j}{\partial z_i} \tilde{g}_k^2, \quad 0 \le j \le n, \quad 1 \le k \le n.$$

Hence

$$\det \left(\frac{\partial^2 \tilde{f}_s}{\partial v_{jk} \partial z_i} - \psi_k \frac{\partial^2 \tilde{f}_s}{\partial u_{kk} \partial z_i} \right)_{i,j=p+1,\ldots,n,} = \tilde{g}_k^{2(m-p)}, \quad k = 1, \ldots, n$$

equals to some n-minors of $d\Phi$. So the zero set $\mathcal{V}(K)$ of the ideal

$$K = \left(\tilde{g}_1^{2(m-p)}, \ldots, \tilde{g}_n^{2(m-p)} \right)$$

contains the critical set of Φ. However,

$$\mathcal{V}(K) \subset \mathcal{V}(\psi^*(g_1^{2(m-p)}), \ldots, \psi^*(g_n^{2(m-p)})) \cap \psi(X^1) = \emptyset.$$

Hence, Φ is a submersion. By [GM] theorem 2.2.3 , there is a dense subset $S_1 \subset S$, such that for any $s \in S_1$, f_s has only A_1 singularities on X^1. Since $j(f) < \infty$ and the theorem 3) in §2.2.7, S_1 is also open.

2) By the theorem in §2.1.7. We can assume that $N = M/T(M)$ is generated by $\bar{g}_{p+1}, \ldots, \bar{g}_n$, define

$$\Delta(z,u,v) = \det \left(h_{kl} + u_{kl} + \sum_{j=0}^{n} z_j v_{jk} \delta_{kl} \right)_{p+1 \le k,l \le n}.$$

Consider the hypersurface $H \subset \mathbb{C}^{n+1} \times S$ defined by $\Delta(z,u,v) = 0$, which is obviuosly reduced(see [P2]), and the projection $p : H \cap \Sigma \longrightarrow S$ induced by the projection from $\mathbb{C} \times S$ to S. Since $\delta_f < \infty$, p is a finite map. Then there exists an analytic subset $B_1 \subset D$ such that $p : p^{-1}(D \setminus B_1) \longrightarrow D \setminus B_1$ is an analytic covering.

Let $S_2 = D \setminus (B_1 \cup \Delta(0, s) = 0)$, which is open and dense in D. For any $s \in S_2$, the hypersurface $\Delta_s(x, y_1, \dots y_n) := \Delta(x, y_1, \dots y_n, u, v) = 0$ intersects $\Sigma \setminus 0$ transversally in δ_f points $D_1, \dots, D_{\delta_f}$ where X is smooth. By [S3] or [P3], f_s has D_∞ singularities in points $D_1, \dots, D_{\delta_f}$. If we take $S' = S_1 \cap S_2$, then we get the theorem. □

Corollary The number of D_∞ points of a good deformation of a function f on $\Sigma \setminus \{0\}$ is δ_f. □

2.2.10 An excellent deformation By the assumptions of the theorem in §2.2.9 and the theorem in §2.1.7, we have $\int \mathfrak{g} = (g_1, \dots, g_p) + (g_{p+1}, \dots, g_n)^2$. For $f \in \int \mathfrak{g}$, we can define another deformation of f. Write

$$f = \sum_{i=1}^{p} a_i g_i + \sum_{i,j=p+1}^{n} h_{ij} g_i g_j.$$

Let

$$f_s = f + \sum_{i=1}^{p} t_i g_i + \sum_{l,k=p+1}^{n} \left(u_{kl} + \sum_{j=0}^{n} z_j v_{jk} \delta_{kl} \right) g_k g_l.$$

We call the above f_s to be the *excellent* deformation of f, since even if $f \in \mathfrak{g}^2$, we can find a $f_s \in \int \mathfrak{g}$ as the “central type”.

Note that we can assume in (2.1.5.1) that $H_i \in (g_{p+1}, \dots, g_n)^2$. Let

$$H_i = \sum_{k,l=p+1}^{n} H^i_{kl} g_k g_l.$$

Define

$$\delta^* := \dim_{\mathbb{C}} \frac{\mathcal{O}}{\det(bh_{kl} - \sum_{i=1}^{p} a_i H^i_{kl}) + \mathfrak{g}}.$$

Then we can prove the following theorem in a similar way to the one in §2.2.9.

Theorem Under the assumptions in theorem 2.2.9. If N is free, $j(f) < \infty$ and $\delta^* < \infty$, then there is an open dense subset S' of $S := \{s = ((t_i), (u_{kl}), (v_{jk})) \mid u_{kl} = u_{lk}\}$ such that for all $s \in S'$,

1) On $X \setminus \Sigma$, f_s has only A_1 critical points;

2) On $X \setminus \{0\}$, f_s has only A_∞ and D_∞ critical points;

3) δ^* is the number of D_∞ points on $\Sigma \setminus \{0\}$. □

2.2.11 Remark For our purpose, a deformation f_s of a function germ f should at least have the following property (S):

(S): There exists an open dense subset S' in the parameter space such that for any $s \in S'$ small enough, f_s has only isolated critical points of type A_1 on $X \setminus \Sigma$, and on $\Sigma \setminus \{0\}$, f_s has only A_∞ and D_∞ singularities.

Chapter 3

The Topology of Non-isolated Singularities

3.1 Topological Properties of the Milnor Fibre

For an analytic function germ $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$, Milnor[Mi] first studied the fibre F of f, called Milnor fibre, and proved, among other things, the homotopy equivalence $F \simeq S^n \vee \cdots \vee S^n$ (μ copies of the sphere) if f defines an isolated singularity. This is the so called bouquet decomposition of the Milnor fibre. Since the real dimension of F is $2n$, this is also called a bouquet of spheres of middle dimension.

After Milnor a lot of authors have been trying to generalize these beautiful results. One direction is to study the Milnor fibre of an analytic function $f : X \longrightarrow \mathbb{C}$ on an analytic space X. Hamm[Ha] has proved that when X is an *icis* and f defines an isolated singularity, then F is (up to homotopy) a bouquet of spheres of middle dimension. Lê has proved the existence of Milnor fibration for any analytic function on any space in [Lê6]. He has also proved that F is again a bouquet of spheres of middle dimension when f defines an isolated singularity and X is a complete intersection or has maximal rectified homotopical depth in [Lê1] and [Lê4]. If f and X both have isolated singularity, Siersma[S6] has proved that F is a wedge of the complex link of X and a bouquet of spheres of middle dimension. Tibăr[Ti1] has given a complete answer to this problem by showing a more general bouquet decomposition of F for f defining an isolated singularity on any analytic space.

At almost the same time, some authors began to consider the case where f defines non-isolated singularities on X. Iomdin[Io] has studied the Euler characteristic of F. When X is smooth, Siersma[S2,3 and 4] has proved that

F is again a bouquet of spheres, but the spheres may have different dimensions, when the singular locus of f is an *icis* or a plane curve and the transversal type is A_1. T. de Jong[deJ] has obtained a similar results when the singular locus of f is a line and the transversal type is A_2, A_3, D_4, E_6, E_7, or E_8. A. Zaharia [Z2] has studied the topology of the Milnor fibre of a function with singular locus a two dimensional *icis* on a smooth space.

In this section, we study the topology of the Milnor fibre of a function defining non-isolated singularities on a singular space. This is motivated by the work of Siersma.

We start with a special case of Lê's theorem on the existence of Milnor fibration, and include the proof since this also helps to fix some notations which will be used in the sequel.

3.1.1 Theorem Let X be a reduced complex analytic space with isolated singularity in x_0. Let Σ be a curve on X with isolated singularity in x_0. Let $f : X \longrightarrow \mathbb{C}$ be a complex analytic function, and the singular locus of f: $\Sigma_f = \Sigma$. Assume that a neighborhood $U \subset X$ of x_0 is embedded in $\mathbb{C}^m$. Then there exists an $\epsilon_0 > 0$ such that for any $\epsilon :\ \epsilon_0 \geq \epsilon > 0$, there is an $\eta_\epsilon > 0$ and for any $\eta :\ \eta_\epsilon \geq \eta > 0$, the function f induces a map $\varphi = \varphi(\epsilon, \eta)$ from $\bar{B}_\epsilon(x_0) \cap X \cap f^{-1}(\bar{\Delta}_\eta(f(x_0)))$ onto $\bar{\Delta}_\eta(f(x_0))$ and its restrictions are locally trivial topological fibrations:

$$\begin{aligned}
\varphi_1 : \bar{B}_\epsilon(x_0) \cap X \cap f^{-1}(\Delta^*_\eta(f(x_0))) &\longrightarrow \Delta^*_\eta(f(x_0)),\\
\varphi : \bar{B}_\epsilon(x_0) \cap X \cap f^{-1}(\bar{\Delta}^*_\eta(f(x_0))) &\longrightarrow \bar{\Delta}^*_\eta(f(x_0)),\\
\varphi^0 : B_\epsilon(x_0) \cap X \cap f^{-1}(\Delta^*_\eta(f(x_0))) &\longrightarrow \Delta^*_\eta(f(x_0)),\\
\tilde{\varphi} : \partial\bar{B}_\epsilon(x_0) \cap X \cap f^{-1}(\bar{\Delta}^*_\eta(f(x_0))) &\longrightarrow \bar{\Delta}^*_\eta(f(x_0)),
\end{aligned}$$

where $B_\epsilon(x_0)$ is an open ball of radius ϵ centered at x_0, $\Delta_\eta(f(x_0))$ is an open disk in $\mathbb{C}$ with $f(x_0)$ as its center and η as its radius, $\Delta^*_\eta(f(x_0)) = \Delta_\eta(f(x_0)) \setminus \{f(x)\}$, and $\bar{\Delta}^*_\eta(f(x_0)) = \bar{\Delta}_\eta(f(x_0)) \setminus \{f(x)\}$.

Proof By the definition of a complex analytic space, there exists a local embedding of a neighborhood U of x_0 into $\mathbb{C}^m$. We assume that the local embedding makes $x_0 = 0$ and $f(x_0) = 0$. Let U be sufficiently small such that $f|_{(X\cap U)\setminus\Sigma}$ has rank 1. In the following we still denote $X \cap U$ by X. Stratify X by $\{X_0 = \{0\}, X_1 = \Sigma \setminus \{0\}, X_2 = X \setminus \Sigma\}$ which is a Whitney stratification of X (see [LT1, 2]).

By [Mi]2.8 and 2.9 (see also [LT1, 2]), there exists a sufficiently small $\epsilon_0 > 0$ such that for any $\epsilon :\ \epsilon_0 \geq \epsilon > 0$, the boundary $\partial\bar{B}_\epsilon$ of the small ball $\bar{B}_\epsilon$ (centered at 0) intersects transversally the strata of $\{X_i\}$.

The Whitney stratification $\{X_i\}$ of X induces a Whitney stratification of $\bar{B}_\epsilon(x_0) \cap X$ whose strata are :

$$B_\epsilon \cap X_0, \quad B_\epsilon \cap X_1, \quad B_\epsilon \cap X_2, \quad \partial\bar{B}_\epsilon \cap X_1, \quad \partial\bar{B}_\epsilon \cap X_2.$$

Since f has at most on $X \setminus \{0\}$ finite number of critical values (one of them is 0), there exists an $\eta_\epsilon > 0$ such that for all $\eta :\ \eta_\epsilon \geq \eta > 0$, the restriction of f to $\partial\bar{B}_\epsilon \cap X_2 \cap f^{-1}(\bar{\Delta}_\eta)$ has real rank 2 in every point, where $\bar{\Delta}_\eta$ is the closure of an open disk $\Delta_\eta \subset \mathbb{C}$ with origin as its center and η as its radius. The surjective map

$$\varphi := \varphi(\epsilon, \eta) : \bar{B}_\epsilon \cap X \cap f^{-1}(\bar{\Delta}_\eta) \longrightarrow \bar{\Delta}_\eta$$

induced by f is therefore stratified (see [LT1] [GM]): The restrictions to $B_\epsilon \cap X_0, B_\epsilon \cap X_1$ and $\partial\bar{B}_\epsilon \cap X_1$, of f are constant, therefore the restrictions are submersive. Moreover, φ is proper. By Thom-Mather first isotopy lemma (see *e.g.* [LT1](1.4.3)), we have the following locally trivial topological fibrations:

$$\varphi_1 : \bar{B}_\epsilon \cap X \cap f^{-1}(\Delta^*_\eta) \longrightarrow \Delta^*_\eta,$$
$$\varphi_2 : \bar{B}_\epsilon \cap X \cap f^{-1}(\partial\bar{\Delta}_\eta) \longrightarrow \partial\bar{\Delta}_\eta.$$

Thus there is a locally trivial topological fibration:

$$\varphi : \bar{B}_\epsilon \cap X \cap f^{-1}(\bar{\Delta}^*_\eta) \longrightarrow \bar{\Delta}^*_\eta.$$

The function f also induces a stratified, proper map from $\partial\bar{B}_\epsilon \cap X \cap f^{-1}(\bar{\Delta}_\eta)$ onto $\bar{\Delta}_\eta$. By Thom-Mather first isotopy lemma (*loc.cit.*), f induces local trivial topological fibrations:

$$\tilde{\varphi}_1 : \partial\bar{B}_\epsilon \cap X \cap f^{-1}(\Delta^*_\eta) \longrightarrow \Delta^*_\eta,$$
$$\tilde{\varphi}_2 : \partial\bar{B}_\epsilon \cap X \cap f^{-1}(\partial\bar{\Delta}_\eta) \longrightarrow \partial\bar{\Delta}_\eta.$$

Hence we have a local trivial topological fibration

$$\tilde{\varphi} : \partial\bar{B}_\epsilon \cap X \cap f^{-1}(\bar{\Delta}^*_\eta) \longrightarrow \bar{\Delta}^*_\eta.$$

Since $\tilde{\varphi}_1$ is a subfibration of φ_1, the restriction of φ_1 gives a local trivial topological fibration:

$$\varphi^0 : B_\epsilon \cap X \cap f^{-1}(\Delta^*_\eta) \longrightarrow \Delta^*_\eta. \qquad \square$$

3.1.2 Theorem Let f_s be a deformation of f satisfying (S) in §2.2.11. Let η, ϵ be the same as in §3.1.1. Then for $\mid s \mid, \eta$ and ϵ sufficiently small, the map

$$f_s : \bar{B}_\epsilon \cap f_s^{-1}(\bar{\Delta}_\eta) \longrightarrow \bar{\Delta}_\eta$$

has the following properties:

1) For all $t \in \bar{\Delta}_\eta$, $f_s^{-1}(t) \bar{\pitchfork} \partial \bar{B}_\epsilon$ (as stratified spaces);

2) For every $t \in \partial \bar{\Delta}_\eta$, and hence for every $t \in \bar{\Delta}_\eta \setminus \{\text{critical values of } f_s\}$, there is a homeomorphism: $f^{-1}(t) \cong f_s^{-1}(t)$;

3) There is a homeomorphism: $f^{-1}(\bar{\Delta}_\eta) \cong f_s^{-1}(\bar{\Delta}_\eta)$. Especially, $f_s^{-1}(\bar{\Delta}_\eta)$ is contractible.

Proof Let S_δ be the intersection of a small open ball in the parameter space with the dense open set which satisfies the condition (S) in §2.2.11. The center of S_δ is 0 and its the radius δ is taken so small that all the A_1-points and all the D_∞-points are contained in $B_{\epsilon/2}$.

1) By the assumption on δ, we know that f_s only has A_∞-points on $\partial \bar{B}_\epsilon$ in which f_s takes 0 as its value. So it is enough to verify that $f_s^{-1}(0) \bar{\pitchfork} \partial \bar{B}_\epsilon$.

For $P \in f_s^{-1}(0) \cap \partial \bar{B}_\epsilon$, if $P \notin \Sigma \cap \partial \bar{B}_\epsilon$, P is not a critical point of f and $f^{-1}(t) \bar{\pitchfork} \partial \bar{B}_\epsilon$, so for small s and t, $f_s^{-1}(t) \bar{\pitchfork} \partial \bar{B}_\epsilon$. If $P \in \Sigma \cap \partial \bar{B}_\epsilon$, P is an A_∞ of f_s. Since $\Sigma \bar{\pitchfork} \partial \bar{B}_\epsilon$, we know that $f_s^{-1}(0) \bar{\pitchfork} \partial \bar{B}_\epsilon$ (see [S3]).

2) Consider the map

$$F(z, s) = (f_s(z), s) : X \times S_\delta \longrightarrow \mathbb{C} \times S_\delta.$$

Define

$$Z = F^{-1}(\bar{\Delta}_\eta \times S_\delta) \cap (\bar{B}_\epsilon \times S_\delta) \quad \text{and} \quad F|_Z : Z \longrightarrow \bar{\Delta}_\eta \times S_\delta.$$

For every $s \in S_\delta$, $F|_Z$ defines a map

$$f_s : f_s^{-1}(\bar{\Delta}_\eta) \cap \bar{B}_\epsilon \longrightarrow \bar{\Delta}_\eta.$$

Let

$$W = F^{-1}(\partial \bar{\Delta}_\eta \times S_\delta) \cap (\bar{B}_\epsilon \times S_\delta)$$

and

$$F|_W : W \longrightarrow \partial \bar{\Delta}_\eta \times S_\delta.$$

Stratify W as follows:

$$W_1 = F^{-1}(\partial \bar{\Delta}_\eta \times S_\delta) \cap (B_\epsilon \times S_\delta), \quad W_2 = F^{-1}(\partial \bar{\Delta}_\eta \times S_\delta) \cap (\partial \bar{B}_\epsilon \times S_\delta).$$

Since for any $s \in S_\delta$, f_s has no critical value on $\partial \bar{\Delta}_\eta$, $F|_W$ is a submersion on W_1. From 1), we know $f_s^{-1}(t) \bar{\pitchfork} \partial \bar{B}_\epsilon$. It follows that $F|_W$ is a submersion on W_2. By a relative version of Ehresmann's fibration theorem (see for example [La] 3.0). $F|_W$ is locally trivial topological fibration over $\partial \bar{\Delta}_\eta \times S_\delta$. Hence

$$f_0^{-1}(t) = F^{-1}(t, 0) \cong F^{-1}(t, s) = f_s^{-1}(t).$$

3) Consider the composition:

$$\pi := p \circ F : F^{-1}(\bar{\Delta}_\eta \times S_\delta) \cap (\bar{B}_\epsilon \times S_\delta) \xrightarrow{F} \bar{\Delta}_\eta \times S_\delta \xrightarrow{p} S_\delta.$$

Stratify $Z = F^{-1}(\bar{\Delta}_\eta \times S_\delta) \cap (\bar{B}_\epsilon \times S_\delta)$ by:
$Z_{1i} = (X_i \times S_\delta) \cap F^{-1}(\Delta_\eta \times S_\delta) \cap (B_\epsilon \times S_\delta)$,
$Z_{2i} = (X_i \times S_\delta) \cap F^{-1}(\Delta_\eta \times S_\delta) \cap (\partial\bar{B}_\epsilon \times S_\delta)$,
$Z_{3i} = (X_i \times S_\delta) \cap F^{-1}(\partial\bar{\Delta}_\eta \times S_\delta) \cap (B_\epsilon \times S_\delta)$,
$Z_{4i} = (X_i \times S_\delta) \cap F^{-1}(\partial\bar{\Delta}_\eta \times S_\delta) \cap (\partial\bar{B}_\epsilon \times S_\delta)$,
where X_i's are defined in the same way as in the proof of the theorem in §3.1.1 (i=0,1,2). Only seven of the Z_{ij}'s are not empty. Since $\{X_i\}$ is a Whitney stratification and for any $t \in \bar{\Delta}_\eta$

$$X_i \times \{t\} \bar{\pitchfork} \partial\bar{B}_\epsilon \times \{t\} \text{ and } X_i \times \{t\} \bar{\pitchfork} (F^{-1}(\partial\bar{\Delta}_\eta \times \{t\}) \cap (\partial\bar{B}_\epsilon \times \{t\})).$$

The stratification of Z given above is also a Whitney stratification. And the restrictions of π to the non-empty strata :

$$\begin{aligned}
&\pi\,|_{Z_{12}}: Z_{12} = \{\ (z,s) \in ((X \setminus \Sigma) \cap B_\epsilon) \times S_\delta) \mid (f_s(z), s) \in \Delta_\eta \times S_\delta\} \longrightarrow S_\delta,\\
&\pi\,|_{Z_{22}}: Z_{22} = \left\{\ (z,s) \in ((X \setminus \Sigma) \cap \partial\bar{B}_\epsilon) \times S_\delta) \mid (f_s(z), s) \in \Delta_\eta \times S_\delta\right\} \longrightarrow S_\delta,\\
&\pi\,|_{Z_{32}}: Z_{32} = \left\{\ (z,s) \in ((X \setminus \Sigma) \cap B_\epsilon) \times S_\delta) \mid f_s(z) \in \partial\bar{\Delta}_\eta\right\} \longrightarrow S_\delta,\\
&\pi\,|_{Z_{42}}: Z_{42} = \left\{\ (z,s) \in ((X \setminus \Sigma) \cap \partial\bar{B}_\epsilon) \times S_\delta) \mid f_s(z) \in \partial\bar{\Delta}_\eta\right\} \longrightarrow S_\delta,\\
&\pi\,|_{Z_{11}}: Z_{11} = \{\ (z,s) \in ((\Sigma \setminus 0) \cap B_\epsilon) \times S_\delta \mid f_s(z) \equiv 0\} \longrightarrow S_\delta,\\
&\pi\,|_{Z_{21}}: Z_{21} = \left\{\ (z,s) \in ((\Sigma \setminus 0) \cap \partial\bar{B}_\epsilon) \times S_\delta) \mid f_s(z) \equiv 0\ \right\} \longrightarrow S_\delta, \text{ and}\\
&\pi\,|_{Z_{10}}: Z_{10} = \{(0,s) \mid s \in S_\delta\} \longrightarrow S_\delta
\end{aligned}$$

are all submersions. Hence the proper map π is stratified, it follows from Thom-Mather first isotopy lemma that π is a locally trivial topological fibration over S_δ. Then we have

$$f_0^{-1}(\bar{\Delta}_\eta) \cong \pi^{-1}(0) \cong \pi^{-1}(s) \cong f_s^{-1}(\bar{\Delta}_\eta). \qquad \square$$

3.1.3 The additivity of the vanishing homology In studying the Milnor fibre of an analytic function, Siersma ([S3-5]) has introduced the homology direct sum decomposition, the *additivity of the vanishing homology,* for functions with non-isolated singularities on a smooth space and functions with isolated singularities on a singular space with isolated singularities. Tibăr [Ti] has observed that this homology direct sum decomposition is also true for an analytic function defining an isolated singularity on any analytic space. In fact, this kind of additivity is also true in our case as indicated by Siersma in [S6] 2.4.

Usually the Milnor fibration is considered in a small neighborhood of the interesting point. In this and the following two subsections, we study the fibration defined by a function in a little bit global situation. The prototype of the function is the deformed function. And once we choose a ball around the interesting point, we will not shrink it any more.

Let X be an analytic space with pure dimension $n+1$ $(n \geq 1)$ embedded in $\mathbb{C}^m$ and $X_{\text{sing}} = \{0\}$. Let $\bar{B}_\epsilon$ be a closed ball in $\mathbb{C}^m$ with radius ϵ and center 0, $\bar{\Delta}_\eta$ a closed disc in $\mathbb{C}$ with radius η and center 0. Let $f : X \longrightarrow \mathbb{C}$ be an analytic function such that $f(0) = 0$, and the critical values: $b_0 = 0, b_1, \ldots, b_\sigma$ of f are all contained in the open disc Δ_η. Assume that all the choices satisfy

$$\boxed{(*) \qquad f^{-1}(t)\bar{\pitchfork}(\partial \bar{B}_\epsilon \cap X) \quad \text{for all } t \in \bar{\Delta}_\eta}$$

(when $t = b_i$, this is considered as stratified set)

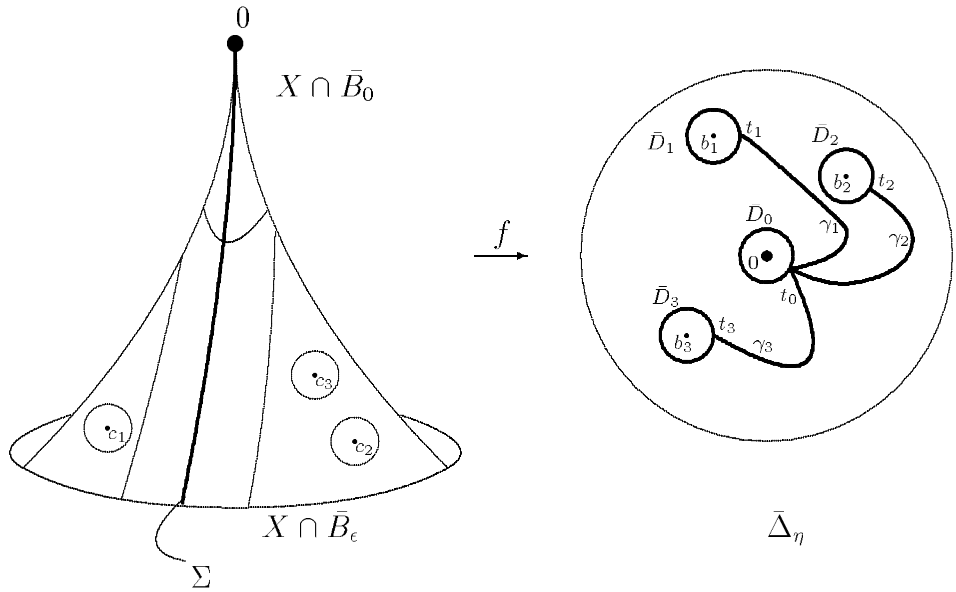


Figure 6

We choose (cf. [S3-5]):
1) a system of disjoint closed disc $\bar{D}_i$ around every critical value b_i;
2) points $t_i \in \partial \bar{D}_i, i = 0, 1, \ldots \sigma$;

3) a system of paths $\gamma_1, \dots, \gamma_\sigma$ from t_i to t_0 in a usual way (see Figure 6);
For any subset $A \subset \mathbb{C}$ and $w \in \mathbb{C}$, we take the following notation:

$$X_A = X \cap \bar{B}_\epsilon \cap f^{-1}(A), \qquad X_w = X \cap \bar{B}_\epsilon \cap f^{-1}(w).$$

4) Denote still by f the restrictions of f :

$$f : (X_{\bar{D}_i}, X_{t_i}) \longrightarrow (\bar{D}_i, t_i).$$

Let Σ_i be the critical locus of f inside $X_{\bar{D}_i}$. Let $r_i : X \longrightarrow \mathbb{R}^+$ be an analytic function defined in a neighbourhood of Σ_i with

$$r_i(z) = 0 \quad \Longleftrightarrow \quad z \in \Sigma_i.$$

Let

$$T_i(\epsilon) = \left\{ z \in \bar{B}_\epsilon \cap X \mid r_i(z) \leq \epsilon \right\}.$$

5) Choose $\epsilon_1 > 0$ such that for any $\epsilon' : 0 < \epsilon' < \epsilon_1$, there exists a $\tau = \tau(\epsilon')$ such that (see the proof in §3.1.8 or [S5],[Mi] or [Io]) $X_{b_i} \bar{\pitchfork} \partial T_i(\epsilon')$ and

$$X_t \bar{\pitchfork} \partial T_i(\epsilon') \quad \text{for all } t \in \bar{D}_i^* = \{ t \in \mathbb{C} \mid 0 < |t - b_i| \leq \tau \}.$$

Let ϵ' be so small that all $T_i(\epsilon')$'s and $\bar{D}_i$'s are disjoint inside $\bar{B}_\epsilon$ and $\bar{\Delta}_\eta$ respectively.

Denote

$$T_i = T_i(\epsilon'),\ E^i = T_i \cap X_{\bar{D}_i},\ F^i = T_i \cap X_{t_i},\ E = X_{\bar{\Delta}_\eta},\ F = X_{t_0} = f^{-1}(t_0) \cap \bar{B}_\epsilon.$$

With appropriate deformation retractions and excisions, similar to [S4 and 5], one can prove the following homology direct sum decomposition formula (all homology groups in this section have integral coefficients):

Theorem (*Additivity of the Vanishing Homology*) Let f be a function on X. Notations and assumptions on f are the same as above then we have

$$H_*(E, F) \cong \bigoplus_{i=0}^{\sigma} H_*(E^i, F^i). \qquad \square$$

3.1.4 Homology bouquet decompositions If one checks the proof of the additivity of the vanishing homology, one can obtain something more precise under some extra assumptions. First we have a corollary from the additivity of the vanishing homology.

Corollary If E and E^i's are contractible, then we have a homology bouquet decomposition

$$\tilde{H}_*(F) \cong \bigoplus_{i=0}^{\sigma} \tilde{H}_*(F^i). \qquad \square$$

Lemma If E and E^i's are contractible, then we have

$$H_*(F, F^0) \cong \bigoplus_{i=1}^{\sigma} \tilde{H}_*(F^i) \tag{3.1.4.1}$$

Proof Consider the exact homology sequence of the triple $(X_{\bar{\Delta}_\eta}, X_{\bar{D}_0}, X_{t_0})$:

$$\cdots \longrightarrow H_q(X_{\bar{D}_0}, X_{t_0}) \xrightarrow{i_q} H_q(X_{\bar{\Delta}_\eta}, X_{t_0}) \xrightarrow{j_q} H_q(X_{\bar{\Delta}_\eta}, X_{\bar{D}_0})$$
$$\xrightarrow{\partial} H_{q-1}(X_{\bar{D}_0}, X_{t_0}) \longrightarrow \cdots$$

By homotopy lifting property and excision, this becomes

$$\cdots \longrightarrow H_q(X_{\bar{D}_0}, X_{t_0}) \xrightarrow{i_q} \bigoplus_{i=0}^{\sigma} H_q(X_{\bar{D}_i}, X_{t_i}) \xrightarrow{j_q} \bigoplus_{i=1}^{\sigma} H_q(X_{\bar{D}_i}, X_{t_i})$$
$$\xrightarrow{\partial} H_{q-1}(X_{\bar{D}_0}, X_{t_0}) \longrightarrow \cdots \tag{3.1.4.2}$$

where i_q is the inclusion to the first factor (see [S6]). Since $(E^i \cup X_{t_i}, X_{t_i})$ and $(X_{\bar{D}_i}, X_{t_i})$ are homotopy equivalent relative to X_{t_i}, by taking excision again ($loc.cit$),

$$\cdots \longrightarrow H_q(E^0, F^0) \xrightarrow{i_q} \bigoplus_{i=0}^{\sigma} H_q(E^i, F^i) \xrightarrow{j_q} \bigoplus_{i=1}^{\sigma} H_q(E^i, F^i)$$
$$\xrightarrow{\partial} H_{q-1}(E^0, F^0) \longrightarrow \cdots \tag{3.1.4.3}$$

Since E^i's are contractible, we have

$$\cdots \longrightarrow \tilde{H}_q(F^0) \xrightarrow{i_q} \bigoplus_{i=0}^{\sigma} \tilde{H}_q(F^i) \xrightarrow{j_q} \bigoplus_{i=1}^{\sigma} \tilde{H}_q(F^i) \xrightarrow{\partial} \tilde{H}_{q-1}(F^0) \longrightarrow \cdots \tag{3.1.4.4}$$

where i_q is the inclusion to the first factor (here we have written i_q, j_q for i_{q+1}, j_{q+1} by abusing the notations).

We have the exact sequence of the pair (F, F^0):

$$\cdots \longrightarrow \tilde{H}_q(F^0) \xrightarrow{i_q} \tilde{H}_q(F) \longrightarrow H_q(F, F^0) \longrightarrow \tilde{H}_{q-1}(F^0) \longrightarrow \cdots$$

By the corollary above and the fact that i_q is an injection, we have the lemma. $\square$

In the remainder of this subsection we consider a more special case:

Let Σ_0 be a curve containing 0. Let $\Sigma_1 = \{c_1\}, \ldots, \Sigma_\sigma = \{c_\sigma\}$ be isolated critical points of f and all these c_j's be contained in the smooth part of X. Assume $X \setminus \Sigma_0$ is smooth, then (see [Mi], [S3, 4 and 6]) for each $i = 1, \ldots, \sigma$:

$$\tilde{H}_q(F^i) = \begin{cases} \mathbb{Z}^{\nu_i}, & q = n, \\ 0, & q \neq n. \end{cases}$$

where ν_i is the Milnor number of f at c_i.

By the corollary above, we have

$$\tilde{H}_q(F) = \begin{cases} \tilde{H}_n(F^0) \oplus \mathbb{Z}^{\nu}, & q = n, \\ \tilde{H}_q(F^0), & q \neq n, \end{cases} \tag{3.1.4.5}$$

where $\nu = \sum\limits_{i=1}^{\sigma} \nu_i$.

And (3.1.4.1) becomes

$$H_q(F, F^0) = \begin{cases} \mathbb{Z}^{\nu}, & q = n, \\ 0, & q \neq n, \end{cases} \tag{3.1.4.6}$$

since i_q is an injection.

It is well known that for each $i > 0$,

$$F^i \quad \overset{h}{\simeq} \quad S^n \vee \cdots \vee S^n \quad (\text{ wedge of } \nu_i \text{ copies of the sphere } S^n).$$

These spheres are vanishing cycles, the corresponding $(n+1)$-cells $e_1, \ldots, e_{\nu_i}$ are the thimbles of the vaninshing cycles. Denote

$$e^i = e_1 \cup \cdots \cup e_{\nu_i}, \quad e = \bigcup_{i=1}^{\sigma} e^i, \quad E^+ = X_{D \cup \Gamma} \quad \overset{h}{\simeq} \quad E, \quad F^+ = X_\Gamma \quad \overset{h}{\simeq} \quad F.$$

By attaching these $(n+1)$-cells along the vanishing cycles, we get $F^+ \cup e$ and the inclusion

$$F^0 \hookrightarrow F^+ \cup e.$$

In the similar way to [S6] (2.6, 2.7, 2.8), we get a homology bouquet decomposition:

Proposition Assume (*) in §3.1.3. If Σ_0 is a curve containing 0, $\Sigma_i = \{c_i\} \subset X_{\mathrm{reg}} \quad (i \geq 1)$, and $X \setminus \Sigma_0$ is smooth, then

$$\begin{cases} H_*(F^0 \vee S) \cong H_*(F) \cong H_*(F^+), \\ H_*(F^0) \cong H_*(F^+ \cup e) \end{cases} \tag{3.1.4.7}$$

where $S = \bigvee_{i=1}^{\nu} S^n$.□

3.1.5 A Homotopy bouquet decomposition In §3.1.4, we, by using mainly homology excision, proved a homology bouquet decomposition of the fibre F (see (3.1.4.7)). In this subsection, we are going to prove a homotopy bouquet decomposition for the fibre F under certain assumptions. We want to use Whitehead's theorem. Some preparations have been done in the last part of §3.1.4.

Recall that F^0 is the fibre of

$$f : T_0 \cap X \cap f^{-1}(\bar{D}_0^*) \longrightarrow \bar{D}_0^* \tag{3.1.5.1}$$

It has been embedded in F. The embedding induces a map $i : F^0 \vee S \longrightarrow F$.

Proposition Let X be complex analytic space of pure dimension $n+1 \geq 4$. Let $f : X \longrightarrow \mathbb{C}$ be an analytic function. Let $\Sigma_0, \Sigma_1 = \{c_1\}, \ldots, \Sigma_\sigma = \{c_\sigma\}$ be the components of the critical locus of f with critical values: $b_0 = 0, b_1, \ldots, b_\sigma$, which are all contained in the interior of Δ_η and $X \setminus \Sigma_0$ is smooth. Let f, $\bar{B}_\epsilon$ and $\bar{\Delta}_\eta$ satisfy (*) in §3.1.3. Let the Σ_0 be at most one dimensional. If F^0 is simply connected, then we have a homotopy bouquet decomposition of F:

$$F \overset{h}{\simeq} F^0 \vee S^n \vee \cdots \vee S^n \tag{3.1.5.2}$$

where the number of the spheres is the sum of the Milnor numbers of f restricted to some neighborhood of c_i for $i = 1, \ldots, \sigma$.

Proof By (3.1.4.6) and relative Hurewicz isomorphism theorem, we have

$$\pi_q(F^+, F^0) = 0 \text{ for } q < n, \quad \pi_n(F^+, F^0) \cong H_n(F^+, F^0).$$

Hence in case $n > 2$, we have

$$\pi_1(F^+) \cong \pi_1(F^0),$$

so

$$\pi_1(F^+) \cong \pi_1(F^0 \vee S).$$

So if F^0 is simply connected, then $F^0 \vee S$ and F are also simply connected, and i induces the isomorphism on homology in (3.1.4.7).

By repeating the arguments of Siersma ([S6]§4), and Whitehead theorem (*loc. cit*, and [Wh]) we can conclude the proof. □

Remarks 1) Let X be smooth. When f is a Morsification of a function with isolated singularity, this theorem is a version of Milnor's bouquet theorem [Mil]. When f is a deformation of a function with isolated line singularity or singular locus a plane curve, the theorem is Siersma's bouquet theorem (see [S3 , 4]).

2) Let f have isolated singularity on X. By adding some generic linear form l to f, one gets a generic approximation $f_s = f + sl$ of f. Then f_s and f have equivalent Milnor fibrations. If X has isolated singularity at 0. Then f_s has only Morse points on the smooth part of X. In this case, the theorem is part of [S6] where the statement is proved for $n \neq 2$.

3.1.6 In the remainder of this section, we consider the following cases: X is an *icis* with pure dimension $(n+1)$ embedded in $\mathbb{C}^m$ such that the singular point is the origin 0. $\Sigma \subset X$ is a one dimensional *icis* in $\mathbb{C}^m$ with 0 as the singular point. Let analytic function germ $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ satisfy the condition $j(f) < \infty$, that is, f is a transversal A_1 singularity along Σ.

Suppose there is a good (or excellent) deformation $f_s : (X \times S, 0) \longrightarrow (\mathbb{C}, 0)$ of $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ satisfying the property (S) in §2.2.11. And for $|s| > 0$ small enough, the map, induced by f_s and still denoted by f_s,

$$f_s : X_{\bar{\Delta}_\eta} = X \cap \bar{B}_\epsilon \cap f_s^{-1}(\bar{\Delta}_\eta) \longrightarrow \bar{\Delta}_\eta$$

satisfies the property (*) in §3.1.3.

Proposition Under the assumptions above, if $F^0 = f^{-1}(0) \cap T \cap \bar{B}_\epsilon$ is simply connected, then we have a bouquet decomposition of the Milnor fibre:

$$F \overset{h}{\simeq} F^0 \vee S^n \vee \cdots \vee S^n,$$

the number of the spheres is $\sharp A_1$ of f_s.

Proof The critical locus of f_s consists of :

- the one dimensional reduced curve Σ, on which f_s has only local A_∞ or D_∞ type singularities off the origin;
- isolated points $c_1, \ldots, c_\sigma$ (we call them Morse or A_1 points), at each c_i, f_s is locally a Morse function.

We suppose that all the critical values of f_s are different for the sake of notation. The critical value $b_0 = 0$ corresponds to the non-isolated singular locus Σ.

Let $\bar{B}_0, \bar{B}_1, \ldots, \bar{B}_\sigma$ be disjoint $2m$-closed balls around $0, c_1, \ldots, c_\sigma$ respectively. Note that one can take $T_i(\epsilon') = \bar{B}_i$ in §3.1.3 for $i = 1, \ldots, \sigma$. The radii are small enough to make sure that the $\bar{B}_i$'s are disjoint inside $\bar{B}_\epsilon$ and all the D_∞ and A_1 points of f_s are contained in $B_\epsilon \setminus \bar{B}_0$.

We choose the above objects in such a way that the maps induced by f_s (still denoted by f_s):

$$f_s : \bar{B}_i \cap f_s^{-1}(\bar{D}_i) \longrightarrow \bar{D}_i \qquad i = 0, 1, \ldots, \sigma$$

are Milnor fibrations over the punctured closed disks $\bar{D}_i^* = \bar{D}_i - b_i$.

Let $\epsilon' : 0 < \epsilon' < \epsilon_1$ be so small that $T := T_0(\epsilon') = \{z \in X \cap \bar{B}_\epsilon \mid r(z) \leq \epsilon'\}$ does not contain any Morse points of f_s (see §3.1.3 4), 5) and figure 6). (3.1.4.5) is

$$\tilde{H}_k(F) = \begin{cases} \tilde{H}_n(F^0) \oplus \mathbb{Z}^\sigma, & k = n, \\ \tilde{H}_k(F^0), & k \neq n. \end{cases}$$

Since we assume F^0 is simply connected, by the proposition in §3.1.5, we have the conclusion. □

Remark In §3.1.12, we will prove a sufficient condition for F^0 to be simply connected: $n \geq 3$, X and $X \cap f^{-1}(0)$ to be complete intersection. The $n = 2$ case is unknown. For the case of $n = 1$, we consider the following example.

Example Let $X = \{(x, y, z) \in \mathbb{C}^3 \mid h = xy + z^2 = 0\}$, which has an A_1 singularity in $\mathbb{C}^3$. Let $\mathfrak{g} = (xy, z)$, which defines a plane curve Σ on X. All the functions having Σ in their singular locus form the primitive ideal $\int \mathfrak{g} = (xy, z^2)$. Consider function $f = xy \in \int \mathfrak{g}$, which is stable in the sense of §1.5. f has one critical value 0 corresponding to the singular locus Σ. The Milnor fibre of f has the homotopy type of two disjoint one-cycles.

When Σ is smooth and $\dim X = 2$, we have a discussion in §3.1.13.

3.1.7 The situation near the singular set Σ We consider next the situation on Σ but outside B_0. Let $\Sigma^* =$ the closure of $(\Sigma - B_0) \cap B_\epsilon$. Since the curve Σ has only isolated singularity at 0, Σ^* is a disjoint union of annuli, the number of them is the number of the irreducible branches of Σ. Denote the annuli by $\Sigma_k^*, \quad k = 1, \ldots, \lambda$.

Let Σ be defined locally by $g_1, \dots, g_n$ inside an open neighborhood of $X \cap \bar{B}_\epsilon$ (see the remark in §2.1.9). Let the r_0 in §3.1.3 4) be defined by

$$r(z) = |g_1(z)|^2 + \cdots + |g_n(z)|^2 : X \longrightarrow \mathbb{R}^+$$

and

$$T = T_0(\epsilon') = \{z \in X \cap \bar{B}_\epsilon \mid r(z) \leq \epsilon'\}, \quad T' := \{z \in X \cap \bar{B}_\epsilon \setminus B_0 \mid r(z) \leq \epsilon'\}.$$

If we take $\epsilon' > 0$ small enough, we have the disjoint union:

$$T^* = \bigsqcup_{k=1}^{\lambda} T_k^*, \quad \partial T^* = \bigsqcup_{k=1}^{\lambda} \partial T_k^*,$$

where T_k^* is the component of T' which contains Σ_k^*, and T_k^*'s do not contains any Morse point of f_s.

$$\partial T_k^* = \{z \in T_k^* \setminus (\partial \bar{B}_\epsilon \cup \partial \bar{B}_0) \mid r(z) = \epsilon'\} \cup (T_k^* \cap \partial \bar{B}_\epsilon) \cup (T_k^* \cap \partial \bar{B}_0).$$

Lemma T^* is diffeomorphic to $\Sigma^* \times Q^n$, where Q^n is a closed ball in $\mathbb{C}^n$.

Proof Define $G : X \longrightarrow \mathbb{C}^n$ by $G(z) = (g_1(z), \dots, g_n(z))$. Restricted to T^*, G is a proper map: $G : T^* \longrightarrow \mathbb{C}^n$ and $G^{-1}(0) = \Sigma^*$. We have known that

$$G^{-1}(0) \bar{\pitchfork} \partial \bar{B}_\epsilon, \quad \text{and} \quad G^{-1}(0) \bar{\pitchfork} \partial \bar{B}_0.$$

And when ϵ' is small enough,

$$G^{-1}(0) \bar{\pitchfork} \{z \in T^* \mid r(z) = \epsilon'\},$$

since 0 is a regular value of G (see [Io] or [Ha]). Since $G^{-1}(0) \cap \partial T^*$ is compact, there exists a neighbourhood U of $0 \in \mathbb{C}^n$ such that for any $u \in U$, $G^{-1}(u) \bar{\pitchfork} \partial T^*$. G is obviously a submersion in the interior of T^*. By relative version of Ehresmann's fibration theorem (see [La] for instance), G is locally a trivial differetiable fibration over an open neighbourhood U of 0 in $\mathbb{C}^n$. We can take U contractable, then this fibration is trivial above U. Take a small closed ball $Q^n \subset U$, we obtain the lemma.□

3.1.8 Lemma There exists an $\epsilon^* > 0$ such that for all $\epsilon' : 0 < \epsilon' < \epsilon^*$
1) $f_s^{-1}(0) \bar{\pitchfork} \partial \bar{B}_0, \qquad f_s^{-1}(0) \bar{\pitchfork} \partial T^*$;
2) There exists a $\tau = \tau(\epsilon')$ such that whenever $0 < |t| \leq \tau$, then

$$f_s^{-1}(t) \bar{\pitchfork} \partial \bar{B}_0, \qquad f_s^{-1}(t) \bar{\pitchfork} \partial T^*.$$

Proof We only need to check the transversality at $P \in \partial T^* \setminus (\partial \bar{B}_0 \cup \partial \bar{B}_\epsilon)$. Since r has finite many critical values when it is restricted to $X \cap f_s^{-1}(0)$, so there exists a $\epsilon^* > 0$ such that for any $\epsilon' : 0 < \epsilon' < \epsilon^*$, $f_s^{-1}(0) \bar{\pitchfork} \partial T^*$. Since $f_s^{-1}(0) \cap \partial T^*$ is compact, for the chosen ϵ', there is a $\tau = \tau(\epsilon') > 0$ such that when $0 < |t| < \tau$, $f_s^{-1}(t) \bar{\pitchfork} \partial T^*$ (see [Io]).□

3.1.9 Along Σ, we have 3 type of singularities: A_∞, D_∞ and central type at the origin. Let f_s' be the restriction of f_s to the transversal slice of Σ off the origin. Then the Milnor fibre pair (F, F') of f_s and f_s' has the following homotopy type:

- at A_∞: $(F, F') \overset{h}{\simeq} (S^{n-1}, S^{n-1})$;
- at D_∞: $(F, F') \overset{h}{\simeq} (S^n, S^{n-1})$;
- *central type*: (F^c, F'), where $F^c = f_s^{-1}(t) \cap X \cap \bar{B}_0$.

3.1.10 Let $\bar{B}_0, T^*$, and D_0 small enough, and $t_0 \in \partial \bar{D}_0$. Denote $F^* = f_s^{-1}(t_0) \cap T^*$, $\quad F_k^* = f_s^{-1}(t_0) \cap T_k^*$, $\quad F^c = f_s^{-1}(t_0) \cap \bar{B}_0$, $F^0 = F^* \cup F^c$.

Inside T^* we use the coordinates $w_0, w_1, \ldots, w_n$, and assume that

$$(w_0, 0, \ldots, 0) \in \Sigma^* \text{ and } (0, w_1, \ldots, w_n) \in Q^n.$$

Consider the projection

$$w_0 : T_k^* \longrightarrow \Sigma_k^* \quad \text{ and } w_0|_{F_k^*} : F_k^* = T_k^* \cap f_s^{-1}(t_0) \longrightarrow \Sigma_k^*,$$

which we can assume to be analytic. $w_0|_{F_k^*}$ is singular at points of $\Gamma \cap \Sigma_k^*$, where Γ is the polar curve of f_s with respect to w_0, and is defined (in local coordinates) by $\frac{\partial f}{\partial w_1} = \cdots = \frac{\partial f}{\partial w_n} = 0$ and it intersects with Σ_k^* exactly at D_∞ points of f_s. Therefore $w_0|_{F_k^*}$ is singular in a neighbourhood of D_∞-points of f_s on Σ_k^*.

Let $S_k^1, \ldots, S_k^{\alpha_k}$ be small disjoint closed discs around the D_∞-points in Σ_k^* (see Figure 7) , where $\alpha_k =^\sharp D_\infty$ on Σ_k^*. Let

$$S_k = \bigcup_{j=1}^{\alpha_k} S_k^j, \quad M_k = \overline{\Sigma_k^* \setminus S_k}.$$

Note that $w_0^{-1}(S_k^j) \cap T_k^* = S_k^j \times Q^n$. We assume the disks $\bar{D}_0, S_k^j$ and the ball Q^n are so small that

$$f_s : (S_k^j \times Q^n) \cap f_s^{-1}(\bar{D}_0) \longrightarrow \bar{D}_0$$

is still a Milnor fibration with respect to the polyball $S_k^j \times Q^n$.

Lemma ([S5]) For the diameter of Q sufficiently small, the projection $w_0|_{F_k^*} : F_k^* \longrightarrow \Sigma_k^*$ is locally trivial above M_k with fibre homotopy equivalent to the Milnor fibre of the quadratic singularities: $w_1^2 + \cdots + w_n^2$.$\square$

3.1.11 Let γ_k^0 be the boundary circle of $S_k^0 := \Sigma_k^* \cap \bar{B}_0$, $s_k^0 \in \gamma_k^0$. Choose a system of paths $\gamma_k^1, \ldots, \gamma_k^{\alpha_k}$ inside M_k from s_k^0 to $s_k^1, \ldots, s_k^{\alpha_k}$ in the usual way (see Figure 7 for the case of $\alpha_k = 3$).

Σ_k^* :

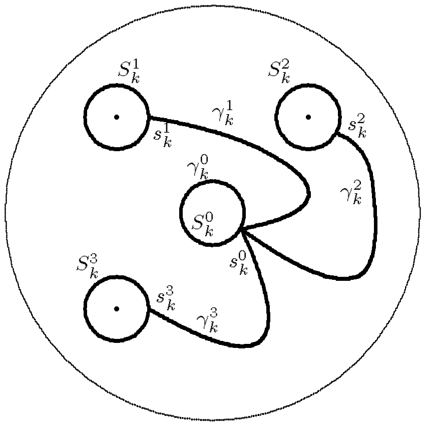


Figure 7

Set $\Gamma_k = \bigcup_{j=0}^{\alpha_k} \gamma_k^j$. Then $S_k \cup \Gamma_k$ is a deformation retract of Σ_k^* and γ_k^0 is a deformation retract of Γ_k. By the local triviality of w_0 and the homotopy lifting property we have

$$(F_k^*, w_0^{-1}(\gamma_k^0)) \quad \overset{h}{\simeq} \quad (w_0^{-1}(S_k \cup \Gamma_k), w_0^{-1}(\Gamma_k)).$$

If $\alpha_k > 0$, then

$$(F_k^*, w_0^{-1}(\gamma_k^0)) \quad \overset{h}{\simeq} \quad (w_0^{-1}(\Gamma_k) \cup E_k, w_0^{-1}(\Gamma_k)),$$

where E_k is a disjoint union of $2\alpha_k$ n-cells attached to the vanishing cycle S^{n-1} in a standard way. The attachment take place in $w_0^{-1}(s_k^0)$.

If $\alpha_k = 0$, then $F_k^* \cup F^c \quad \overset{h}{\simeq} \quad F^c$.

So we have

$$F^0 = F^c \cup F_1^* \cup \cdots \cup F_\lambda^*$$

$$\overset{h}{\simeq} F^c \cup w_0^{-1}(\gamma_1^0) \cup \cdots \cup w_0^{-1}(\gamma_\lambda^0) \cup E_1 \cup \cdots \cup E_\lambda$$
$$\overset{h}{\simeq} F^c \cup E_1 \cup \cdots \cup E_\lambda.$$

The last homotopy equivalence follows from the fact that $w_0^{-1}(\gamma_k^0) \subset F^c$ homotopically and $w_0^{-1}(s_k^0)$ is homotopy equivalent to the Milnor fibre of f'_s, the restriction of f_s to the transversal slice of Σ, and this fibre is of the homotopy type S^{n-1}.

Proposition F^0 can be obtained from F^c by attaching $2^\sharp D_\infty$ n-cells along a transversal vanishing cycle of F^c:

$$F^0 \overset{h}{\simeq} F^c \cup e_1 \cup \cdots \cup e_{2^\sharp D_\infty}. \qquad \square$$

Corollary

1) For the Euler characteristic of F, we have

$$\boxed{\chi(F) = \chi(F^c) + (-1)^n(2^\sharp D_\infty + {}^\sharp A_1)} \qquad (3.1.11.1)$$

2) Especially, we have

$$\boxed{\mu_n - \mu_{n-1} = \mu_n^c - \mu_{n-1}^c + 2^\sharp D_\infty + {}^\sharp A_1} \qquad (3.1.11.2)$$

where $\mu_q = l_{\mathbb{Z}}(H_q(F)), \mu_q^c = l_{\mathbb{Z}}(H_q(F^c))$ are the lengths of the $\mathbb{Z}-$modules, and are called the *q-th Bitti numbers* of the spaces. μ_n is called the *Milnor number* of $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$.

3.1.12 Conclusions The proposition in §3.1.11 can be useful for more precise homology and homotopy results about F. Further investigation about F^c could give some more insight into the homotopy type of F^c and F. In the next subsection, we deal with some special cases where the homotopy type of F^c can be described.

Proposition 1) If F^c is $n-1$ connected ($n \geq 3$), then $F^c \overset{h}{\simeq} S^n \vee \cdots \vee S^n$ (μ_n^c copies), hence $F \overset{h}{\simeq} S^n \vee \cdots \vee S^n$ ($\mu_n^c +^\sharp A_1 + 2^\sharp D_\infty$ copies);

2) If X and $X \cap f^{-1}(0)$ are complete intersections and $X \setminus f^{-1}(0)$ is smooth, then F^0 is $n-2$ connected. If $n \geq 3$, it follows that F^c is simply connected and

$$F \overset{h}{\simeq} F^0 \vee S^n \vee \cdots \vee S^n.$$

Proof 1) is obvious. For 2), Iomdin[Io] proved F^c is $n-2$ connected. Hence F^0 is also $n-2$ connected since the attaching of n-cells does not change the $n-2$ connectedness. The decomposition follows from the proposition in §3.1.6.$\square$

3.1.13 Applications and examples

Corollary 1 Let $X = \mathbb{C}^{n+1}$ $(n \geq 2)$ and $\Sigma = x$-axis. Since F^c is the Milnor fibre of A_∞, $\mu_n^c = 0, \mu_{n-1}^c = 1$, we get a formula of Siersma[S3]:

$$\mu_n = 2^\sharp D_\infty + {}^\sharp A_1 - 1.$$

And when ${}^\sharp D_\infty > 0$ (i.e. f is not of A_∞ type), F is a bouquet of spheres of middle dimension.□

Corollary 2 Let $X = \mathbb{C}^{n+1}$ $(n \geq 2)$ and Σ a plane curve. Since (see [S4]) $F^c \overset{h}{\simeq} S^{n-1} \vee S^n \cdots \vee S^n$, $\mu_n^c = 2\mu(\Sigma)$ (the Milnor number of Σ), $\mu_{n-1}^c = 1$, we have another formula of Siersma (*loc.cit.*)

$$\mu_n = 2\mu(\Sigma) + 2^\sharp D_\infty + {}^\sharp A_1 - 1.$$

When ${}^\sharp D_\infty > 0$, F is a bouquet of spheres of middle dimension. and when ${}^\sharp D_\infty = 0$, $F \overset{h}{\simeq} S^{n-1} \vee S^n \cdots \vee S^n$. □

Remark 1 The conclusions in §3.1.5 are not applicable to the case $\dim X = n + 1 = 2$, *i.e.* $n = 1$. One can check that the proposition in §3.1.4 is valid for $n = 1$, especially, we have

$$H_1(F) = H_1(F^0 \vee S), \quad H_0(F) = H_0(F^0 \vee S) \quad \text{and} \quad S = \bigvee_{i=1}^{\nu} S^1.$$

If F^c is 0-connected, then

$$F^0 \overset{h}{\simeq} F^c \cup e_1 \cup \cdots \cup e_{2^\sharp D_\infty} \tag{3.1.13.1}$$

is also 0-connected, where e_j's are one cells. So $F^0 \vee S$ is 0-connected. Hence F is 0-connected and the Milnor fibre of f is 0-connected.

Formula (3.1.11.2) is still true for $n = 1$. If F^c is 0-connected, then

$$\mu = \mu^c + {}^\sharp A_1 + 2^\sharp D_\infty \tag{3.1.13.2}$$

where μ is the Milnor number of f (the number of circles in F).

Examples for F^c not to be 0-connected are available: $X = \mathbb{C}^2$ and $f = y^2$ is an isolated line singularity with the non-connected Milnor fibre. The example in §3.1.6 is another one to show non-connectedness of the Milnor fibre.

Example 1 We consider the homotopy type of the Milnor fibre of functions with line singularities on a surface with A_k $(k \geq 1)$, D_k $(k \geq 4)$, or E_7 type singularity. According to our theory, we need to study the Milnor fibre of the central type.

Let Σ denote the line on X, the surface with singularity type given above. Σ is defined by the ideal (y,z). Let $f:(X,0)\longrightarrow(\mathbb{C},0)$ be a germ such that $j(f)<\infty$, f_s be the excellent deformation of f. Let F be the Milnor fibre of f, and F^c the central type of F.

Case I: Σ and X have minimal torsion number. Note that the germs considered in the following are weighted homogeneous.

1) If X is the A_1 surface defined by $xy+z^2=0 \quad (A_{1,1})$, then, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y$ (see §1.5 Th. 1.5.3). Hence

$$F^c=\{(x,y,z)\in\mathbb{C}^3 \mid xy+z^2=0, y=1\}.$$

which is contractible (see also [SS] and [ST]). Hence

$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2{}^\sharp D_\infty \text{ copies}).$$

2) If X is an A_k surface defined by $xy+z^{k+1}=0 \quad (A_{k,1} \quad k>1)$, then at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y+z^2$ (see §1.5 Th. 1.5.3). Hence (see also [SS] and [ST])

$$F^c=\{(x,y,z)\in\mathbb{C}^3 \mid xy+z^{k+1}=0, y+z^2=1\}\overset{h}{\simeq}S^1\vee S^1,$$
$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2{}^\sharp D_\infty+2 \text{ copies}).$$

3) If X is a D_k $(k\geq 4)$ surface defined by $x^2y+y^{k-1}+z^2=0 \quad (D_{k,2}$ $k\geq 4)$, then, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y$ (see §1.5.5). Hence

$$F^c=\{(x,y,z)\in\mathbb{C}^3 \mid x^2y+y^{k-1}+z^2=0, y=1\}\overset{h}{\simeq}S^1,$$
$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2{}^\sharp D_\infty+1 \text{ copies}).$$

4) If X is the E_7 surface defined by $x^3y+y^3+z^2=0$, then, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y$ (*loc. cit.*). Hence

$$F^c=\{(x,y,z)\in\mathbb{C}^3 \mid x^3y+y^3+z^2=0, y=1\}\overset{h}{\simeq}S^1\vee S^1,$$
$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2{}^\sharp D_\infty+2 \text{ copies}).$$

Case II: Σ and X have maximal torsion number.

1) If $X: \quad x^l y+y^2+z^2=0 \quad (A_{2l-1,l}, l>1)$, then, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y$ (see §1.5.6). Hence (see [SS] and [ST])

$$F^c=\{(x,y,z)\in\mathbb{C}^3 \mid x^l y+y^2+z^2=0, y=1\}\overset{h}{\simeq}S^1\vee\cdots\vee S^1(l-1\text{ copies}),$$
$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2^\sharp D_\infty+l-1\text{ copies}).$$

2) Let $X: \quad x^l y+yz+xz^2=0 \quad (A_{2l,l}, l>1)$. then, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y$. In order to determine F^c, we change the coordinates by $y'=y+xz$ or $y=y'-xz$ Hence

$$\begin{aligned} F^c &= \{(x,y,z)\in\mathbb{C}^3 \mid x^l y+yz+xz^2=0, y=1\}\\ &= \{(x,y',z)\in\mathbb{C}^3 \mid x^l y'-x^{l+1}z+y'z=0, y'-xz=1\}\\ &= \{(x,y',z)\in\mathbb{C}^3 \mid x^l+y'z=0, y'-xz=1\},\end{aligned}$$

which can be considered as the Milnor fibre of the function $g=y-xz$ on the A_{l-1} surface. Notice that g defines an isolated singularity. By [BR]7.7 (see also [SS] and [ST]),

$$F^c\overset{h}{\simeq}S^1\vee\cdots\vee S^1(l+1\text{ copies}),$$
$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2^\sharp D_\infty+l+1\text{ copies}).$$

3) Let $X: \quad x^l y+xy^2+z^2=0 \quad (D_{2l,l}, l>2)$. Then, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y$. Hence (see also [SS] and [ST])

$$F^c=\{(x,y,z)\in\mathbb{C}^3 \mid x^l y+xy^2+z^2=0, y=1\}\overset{h}{\simeq}S^1\vee\cdots\vee S^1(l-1\text{ copies}),$$
$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2^\sharp D_\infty+l-1\text{ copies}).$$

4) Let $X: \quad x^l y+xz^2+y^2=0 \quad (D^*_{5,2}\text{ or }D_{2l+1,l}, l>2)$. Then, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y$.

$$F^c=\{(x,y,z)\in\mathbb{C}^3 \mid x^l y+xz^2+y^2=0, y=1\}\overset{h}{\simeq}S^1\vee\cdots\vee S^1(l+1\text{ copies}),$$
$$F\overset{h}{\simeq}S^1\vee\cdots\vee S^1 \quad (\mu={}^\sharp A_1+2^\sharp D_\infty+l+1\text{ copies}).$$

Case III: $\varrho(\Sigma,X)$ is not maximal and not minimal. In this case, it is not easy to determine F at this moment. For a complete answer for this case see §3.3. However, in some special cases, we can still give the answer. As an example, we consider f on $A_{5,2}$ surface defined by $x^2y + x^2z^2 + yz = 0$. At the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y + bz^2 \quad (b \neq 1, 0)$. And by [SS],

$$\begin{aligned} F^c &= \{(x, y, z) \in \mathbb{C}^3 \mid x^2y + x^2z^2 + yz = 0, y + bz^2 = 1\} \\ &\overset{h}{\simeq} S^1 \vee \cdots \vee S^1 (6 \text{ copies}), \\ F &\overset{h}{\simeq} S^1 \vee \cdots \vee S^1 \quad (\mu = {}^\sharp A_1 + 2^\sharp D_\infty + 6 \text{ copies}). \end{aligned}$$

Summary In this eaxmple, we have proved that for each function that defines a line singularity on a surface X with $A_1, A_2, A_3, A_4, A_5, D_k$ or E_7 type isolated simple singularity (no conditions on torsion number), the Milnor fibre F of f is a bouquet of circles, and the Milnor number μ of f, the number of the circle in F, is given by the formula (3.1.13.2).

Example 2 One may wonder how is the Milnor fibres of functions with line singularities on higher dimensional hypersurface with isolated simple singularities. For the positions of Σ and X considered in example 1, it is easy to answer by suspension.

Let $X \subset \mathbb{C}^{n+2}$ be defined by $h := xy + z_1^2 + \cdots + z_n^2 = 0$, the A_1 singularity, and $\Sigma = x$-axis, defined by $\mathfrak{g} = (y, z)$. Let $f \in \int_X \mathfrak{g} = (y) + (z)^2$ and $j(f) < \infty$.

Let f_s be a generic deformation of f in $\int_X \mathfrak{g}$ with the coefficient of y nonzero. Then f_s has only Morse points outside Σ, and A_∞ and D_∞ on $\Sigma - \{0\}$. Near the origin, the central type, f_s is $\mathcal{R}_{\Sigma X}$-equivalent to y (see §1.5 Theorem 1.5.4). Hence

$$\begin{aligned} F^c &= \{(x, y, z) \mid xy + z_1^2 + \cdots + z_n^2 = 0, y = 1\} \\ &= \{(x, 1, z) \mid x + z_1^2 + \cdots + z_n^2 = 0\} \end{aligned}$$

which is contractible. Hence $\mu_n^c = \mu_{n-1}^c = 0$, and

$$\mu_n = 2^\sharp D_\infty +^\sharp A_1, \quad \chi(F) = 1(-1)^n(2^\sharp D_\infty +^\sharp A_1).$$

In fact, by §3.1.6 proposition and §3.1.11 Corollary, in this case we can have more:

$$F \overset{h}{\simeq} S^n \cdots \vee S^n.$$

The number of spheres in the wedge is μ_n.

Remark 2 We have seen in §1.4 that there exist two classes of smooth curves on D_5 surface with the same torsion number 2. So the torsion number cannot tell the differences of this two kind of lines. However, if we consider the Milnor fibres of functions on the D_5 surface with different lines as singular loci, we find the differences. Obviously functions $f_1 = y : \{(x,y,z) \mid x^2y+y^4+z^2 = 0\} \longrightarrow \mathbb{C}$ and $f_2 = y : \{(x,y,z) \mid x^2y + xz^2 + y^2 = 0\} \longrightarrow \mathbb{C}$ are stable and their Milnor fibres have the homotopy type of S^1 and $S^1 \vee S^1 \vee S^1$ respectively. (see Example 1 above case I 3) and case II 4))

3.2 A Formula for the Euler Characteristic

For an analytic function $f : X \longrightarrow \mathbb{C}$ defining a non-isolated singularity on X, there is a so called Iomdin-Lê formula (cf. [Lê], [Ti]) which expresses the Euler characteristic of the Milnor fibre of f by that of the *series* of f with isolated singularities. When X is $\mathbb{C}^n$, the singular locus of f is a one dimensional isolated singularity of complete intersection Σ, and the transversal singularity type of f along Σ is Morse, Pellikaan [P4], by considering also series of singularities, has proved a formula for the Euler characteristic of the Milnor fibre of f. This formula expresses the Euler characteristic in terms of the so called Jacobian number $j(f)$, δ and the multiplicity of Σ. These numbers can be computed directly by counting the dimensions of certain finite dimensional vector spaces. Moreover, these numbers have their geometric meaning: δ is the number of D_∞ points in a generic deformation of (f, Σ), and the Jacobian number is the sum of the number of D_∞ points and the number of Morse points in a generic deformation of (f, Σ). The development of computer algebra makes this kind of algebraic formulae more and more important and popular. In this section, we consider the similar question for a function $f : X \longrightarrow \mathbb{C}$ with non-isolated singularities. We do get the similar but not exact the flavor of Pellikaan's. Remark that for a function f with isolated singularity on a weighted homogeneous *icis* X, Bruce and Robert [BR] have proved an algebraic formula for the Milnor number of f (see §3.2.3).

3.2.1 Let $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ be an isolated complete intersection singularity (*icis*) defined by a radical ideal $\mathfrak{h} = (h_1, \ldots, h_p) \subset \mathcal{O}_{\mathbb{C}^{n+1}}$. Let $\Sigma \subset X$ be a one-dimensional *icis* with the origin as the singular point, defined by a radical ideal $\mathfrak{g} \subset \mathcal{O} := \mathcal{O}_{\mathbb{C}^{n+1}}$. There exist *admissible linear forms* l (see [Lê 5]) such that $\{l = 0\} \cap \Sigma = \{0\}$ and $\{l = 0\}$ intersects both X and Σ transversally, and $\{l = 0\} \cap X \cap f^{-1}(0)$ has isolated singularity at the origin. We assume $\{l = 0\}$ is the first coordinate hyperplane of $\mathbb{C}^{n+1}$, and X is weighted homogeneous with respect to the coordinates chosen.

3.2.2 Let D_X be the set of derivations on X, and $\xi_E, \xi^1, \ldots \xi^s$ be the generators of D_X (see [W]), where ξ_E is the Euler derivation. Denote by D_X^0 the submodule of D_X generated by ξ_E and those ξ^i such that if we write $\xi^i = \sum_{i=0}^n \xi_j^i \frac{\partial}{\partial z_j}$, then $\xi_0^i \notin \mathfrak{g}$, and by D_X^1 the submodule of D_X generated by those ξ^i such that if we write $\xi^i = \sum_{i=0}^n \xi_j^i \frac{\partial}{\partial z_j}$, then $\xi_0 \in \mathfrak{g}$, thus $D_X = D_X^0 + D_X^1$. Denote

$$J^0(f) := D_X^0(f) = \{\xi(f) \mid \xi \in D_X^0\}, \quad J^1(f) := D_X^1(f) = \{\xi(f) \mid \xi \in D_X^1\}.$$

If f is clear from the context we just write J^0 and J^1.

Lemma (cf. [P4]) Let $f \in \int \mathfrak{g}$, and $z_0, z_1, \dots, z_n$ the coordinates of $\mathbb{C}^{n+1}$ such that $z_0 = 0$ is admissible. The transversal singularity type of f along every branch of Σ is constant at all the points of $\Sigma \setminus \{0\}$ if and only if

$$\dim_{\mathbb{C}} \left(\frac{\mathcal{O}}{\mathfrak{g} + ((J^1 + \mathfrak{h}) : J^0)} \right) < \infty.$$

Proof The inequality means that $\Sigma \cap \mathcal{V}((J^1 + \mathfrak{h}) : J^0) = \{0\}$. For $\epsilon > 0$ small enough, let $P \in \Sigma \cap \{z_0 = t\}$, $0 < |t| < \epsilon$. $P \notin \mathcal{V}((J^1 + \mathfrak{h}) : J^0)$ if and only if $(J^0)_P \subset (J^1)_P$. Since $z_0 = 0$ intersects both Σ and X transversally and X is smooth at P, we can choose local coordinates such that locally Σ is the first coordinate axis $y = 0$. Furthermore, we can arrange the coordinate transformation such that under this transformation D_X^0 and D_X^1 are preserved. Namely any derivation of X with the first component non-zero at P will remain non-zero at P and any derivation of X with first component zero at P will remain zero at P. We let $x = z_0, y_1, \dots, y_{n-p}$ be the new local coordinates, then at P, $(J^0)_P = \left(\frac{\partial f}{\partial x}\right) \mathcal{O}_{X,P}$ and $(J^1)_P = \left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_{n-p}}\right) \mathcal{O}_{X,P}$. By [P4], this is equivalent to the transversal singularity type of f is constant.$\square$

3.2.3 Let X, Σ and f be the same as before. There is an integer k_0 such that for all $k \geq k_0$, $f_k = f + \frac{1}{k+1} x^{k+1}$ defines an isolated singularity at O. Let $\mu(f_k)$ be the Milnor number of f_k, by [BR] 7.7, we have

$$\mu(f_k) = \dim \left(\frac{\mathcal{O}_{\mathbb{C}^{n+1}}}{\mathfrak{h} + J_X(f_k)} \right).$$

By the exact sequence

$$0 \longrightarrow \frac{\mathfrak{g} + J_X(f_k)}{\mathfrak{h} + J_X(f_k)} \longrightarrow \frac{\mathcal{O}}{\mathfrak{h} + J_X(f_k)} \longrightarrow \frac{\mathcal{O}}{\mathfrak{g} + J_X(f_k)} \longrightarrow 0$$

we know that

$$\mu(f_k) = \dim \left(\frac{\mathfrak{g} + J_X(f_k)}{\mathfrak{h} + J_X(f_k)} \right) + \dim \left(\frac{\mathcal{O}}{\mathfrak{g} + J_X(f_k)} \right) \qquad (3.2.3.1)$$

Define

$$e_k := \dim \frac{\mathcal{O}}{\mathfrak{g} + J_X(f_k)}, \qquad \sigma(\Sigma X, 0) := \dim \left(\frac{\mathcal{O}_\Sigma}{J_X(x)} \right),$$

$$\mathrm{multi}_x(\Sigma) := \dim \left(\frac{\mathcal{O}_\Sigma}{(x)} \right).$$

Then obviously

$$e_k = \sigma(\Sigma X, 0) + k\text{multi}_x(\Sigma)$$

and

$$\frac{\mathfrak{g} + J_X(f_k)}{\mathfrak{h} + J_X(f_k)} = \frac{\mathfrak{g}}{(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g}}.$$

Hence

$$\mu(f_k) = \sigma(\Sigma X, 0) + k\text{multi}_x(\Sigma) + \dim\left(\frac{\mathfrak{g}}{(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g}}\right) \tag{3.2.3.2}$$

We have

$$(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} = ((\xi_E(f_k)) + J^0(f_k)) \cap \mathfrak{g} + J^1(f_k) + \mathfrak{h}.$$

where ξ_E is the Euler derivation and $\xi^0 \in D^0_X$.

3.2.4 In this subsection we assume that Σ is a line in $\mathbb{C}^{n+1}$ defined by the ideal $\mathfrak{g}=(y_1, \ldots, y_n)$. Since X is an *icis* defined by $\mathfrak{h} = (h_1, \ldots, h_p) \subset \mathfrak{g}$, we can write (see §2.1.5)

$$\begin{cases} h_1 = & b_{11}y_1 + \cdots + b_{1p}y_p + h'_1 \\ h_2 = & b_{21}y_1 + \cdots + b_{2p}y_p + h'_2 \\ \cdots & \cdots \\ h_p = & b_{p1}y_1 + \cdots + b_{pp}y_p + h'_p \end{cases}$$

where $h'_1, \ldots, h'_p \in \mathfrak{g}^2$, $y_1, \ldots y_p$ are projected to zero or the generators of the torsion part of the $\mathcal{O}_\Sigma$-module $M = \frac{\mathfrak{g}}{\mathfrak{g}^2+\mathfrak{h}}$ (see §2.1.5) and $y_{p+1}, \ldots y_n$ form a basis of the free part of M since X is a complete intersection. Moreover the determinant b of the matrix $B = (b_{ij})$ is a non-zero divisor in $\mathcal{O}_\Sigma$ and $\bar{b}\bar{y_i} = 0$ in M for $i = 1, \ldots p$.

3.2.5 Lemma If Σ is a line on X, then $D_X = \mathcal{O}\xi_E + D^1_X$ and

$$\begin{aligned} J_X(f_k) &= (\xi_E(f_k))\mathcal{O} + I \\ &\subset (\xi_E(f_k))\mathcal{O} + D^1_X(f) + D^1_X(x^{k+1}), \end{aligned}$$

where $I = \{\xi(f_k) \mid \xi \in D^1_X\} \subset \mathfrak{g}$.

Proof If we write h_i in the form of §3.2.4, it is easy to check that the trivial derivations on X are just those with the coefficient of $\frac{\partial}{\partial x}$ in $\mathfrak{g}$ or b. Since $\mathcal{O}_\Sigma$ is a principal ideal domain and X has isolated singularity at O, $\bar{b} \in (\bar{x})$. Let $\bar{b} = \bar{x}\bar{b}_1$. If ξ^0 is a trivial derivation with b as the coefficient of $\frac{\partial}{\partial x}$, then $\tilde{\xi}^0 = \xi^0 - b_1\xi_E$ has the coefficient of $\frac{\partial}{\partial x}$ in $\mathfrak{g}$. This proves the equality.□

3.2.6 Lemma (cf. [P4]) Let $f \in \int \mathfrak{g}$ have transversal A_1 singularity along the line Σ. For $k > 0$ sufficiently large, we have

$$\tilde{J} := \xi_E(f)\mathfrak{g} + D^1_X(f) + \mathfrak{h} = (\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} \tag{3.2.6.1}$$

Proof By lemma 3.2.5, we have

$$\begin{aligned}(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} &= (\xi_E(f_k)) \cap \mathfrak{g} + I + \mathfrak{h} \\ &\subset (\xi_E(f_k)) \cap \mathfrak{g} + x^k D^1_X(x) + D^1_X(f) + \mathfrak{h} \end{aligned} \tag{3.2.6.2}$$

For $a \in (\xi_E(f_k)) \cap \mathfrak{g}$, $a = a_0 x^{k+1} + a_0 \xi_E(f) \in \mathfrak{g}$. Since $\xi_E(f) \in \mathfrak{g}$, $a_0 \in \mathfrak{g}$. Hence $(\xi_E(f_k)) \cap \mathfrak{g} = (\xi_E(f_k))\mathfrak{g}$ and

$$\begin{aligned}(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} &= (x^{k+1} + \xi_E(f))\mathfrak{g} + I + \mathfrak{h} \\ &\subset (x^{k+1} + \xi_E(f_k))\mathfrak{g} + x^k D^1_X(x) + D^1_X(f) + \mathfrak{h} \end{aligned} \tag{3.2.6.3}$$

Since $j(f) = \dim \frac{\mathfrak{g}}{\mathfrak{h}+J_X(f)} < \infty$, there is a k_1 such that when $k > k_1$, $x^k \mathfrak{g} \subset \mathfrak{h} + J_X(f) = (\xi_E(f)) + D^1_X(f) + \mathfrak{h}$.

By lemma 3.2.2, there is a k_2 such that $x^{k_2} \in \mathfrak{g} + ((D^1_X(f) + \mathfrak{h}) : (\xi_E(f)))$, and $x^{k_2}\xi_E(f) \in (\xi_E(f))\mathfrak{g} + D^1_X(f) + \mathfrak{h}$. Hence when $k > k_1 + k_2$,

$$a_0(x^{k+1} + \xi_E(f)) + b_0 x^k \in \tilde{J} \quad \text{and} \quad (\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} \subset \tilde{J}.$$

On the other hand, there is an integer $n_1 >> 0$ such that when $k \geq n_1$ (see (3.2.3.2))

$$x^k \mathfrak{g} \subset (\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} \subset \tilde{J} \subset \mathfrak{g}.$$

Then for $k > n_1$, by the equality in (3.2.6.3)

$$\begin{aligned}\tilde{J} &\subset \xi_E\left(f + \frac{x^{k+1}}{k+1}\right)\mathfrak{g} + x^{k+1}\mathfrak{g} + I + \mathfrak{h} + x^k D^1_X(x) \\ &= (\xi_E(f) + x^{k+1})\mathfrak{g} + I + \mathfrak{h} + x^k(x\mathfrak{g} + D^1_X(x)) \\ &= (\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} + x^k(x\mathfrak{g} + D^1_X(x)) \\ &\subset (\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} + x^k\mathfrak{g} \\ &\subset (\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} + x^{k-n_1}\tilde{J} \\ &\subset (\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} + \mathfrak{m}\tilde{J}. \end{aligned}$$

By Nakayama's lemma, $(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g} = \tilde{J}$.$\square$

3.2.7 Lemma Under the assumption of 3.2.6, we have

1) $L := \dfrac{\mathfrak{h} + J_X(f)}{\tilde{J}} \cong \dfrac{(\xi_E(f))}{(\xi_E(f)) \cap \tilde{J}}$;

2) $\mathrm{Ann}(L) = \mathfrak{g} + ((D_X^1(f) + \mathfrak{h}) : (\xi_E(f))) = ((D_X^1(f) + \mathfrak{h}) : (\xi_E(f)))$;

3) $L \cong \dfrac{\mathcal{O}}{\mathrm{Ann}(L)} = \dfrac{\mathcal{O}}{((D_X^1(f) + \mathfrak{h}) : (\xi_E(f)))}$.

Proof It is an easy exercise in commutative algebra, we omit it.□

3.2.8 From the exact sequence

$$0 \longrightarrow \frac{\mathfrak{h} + J_X(f)}{(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g}} \longrightarrow \frac{\mathfrak{g}}{(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g}} \longrightarrow \frac{\mathfrak{g}}{\mathfrak{h} + J_X(f)} \longrightarrow 0,$$

we have

$$\mu(f_k) = j(f) + e_k + \dim\left(\frac{\mathfrak{h} + J_X(f)}{(\mathfrak{h} + J_X(f_k)) \cap \mathfrak{g}}\right).$$

Notice that $\sigma(\Sigma X, 0) = \mathrm{multi}_x(\Sigma) = 1$, $e_k = k + 1$. By lemma 3.2.7, we have

$$\mu(f_k) = k + 1 + j(f) + \dim\left(\frac{\mathcal{O}}{((D_X^1(f) + \mathfrak{h}) : (\xi_E(f)))}\right) \qquad (3.2.8.1)$$

Let F and F_k are the Milnor fibre of f and f_k respectively.

Iomdin-Lê's formula ([Lê5], [Ti2])

$$\boxed{\chi(F) = \chi(F_k) + (-1)^{\dim X}(k+1)} \qquad (3.2.8.2)$$

But $\chi(F_k) = 1 + (-1)^{\dim X - 1}\mu(f_k)$. Since in our case $\sigma(\Sigma X, 0) = 1$, we have proved

Proposition For an analytic function f on a weighted homogeneous icis X with singular locus a line Σ and A_1 as transversal singularity type, the Euler characteristic of the Milnor fibre of f is

$$\boxed{\chi(F) = 1 + (-1)^{\dim X - 1}(j(f) + \nu)} \qquad (3.2.8.3)$$

where

$$\nu = \dim\left(\frac{\mathcal{O}}{((D_X^1(f) + \mathfrak{h}) : (\xi_E(f)))}\right). \qquad \square$$

Remark This is an important formula since it allows us to use a computer program (we use *Singular* [GPS]) to compute the Euler characteristic.

3.2.9 Example Let X be the A_1 singularity defined by $h = xy + z^2$. $f : X \longrightarrow \mathbb{C}$ defined by $f = yz + x^k z^2$. Then Σ is the x-axis. $j(f) = 2k + 2$ and

$$((D^1_X(f) + \mathfrak{h}) : \xi_E(f)) = (x^{k+1}, y, z), \quad \nu = k + 1.$$

So $\chi(F) = 1 - (3k + 3)$.

See §1.5 Table 2 and 3 for more examples.

3.2.10 Recall that the torsion number defined in §2.1 is

$$\varrho(\Sigma X) = \dim(T(M)) = \dim\left(\frac{\mathcal{O}_\Sigma}{(\bar{b})}\right).$$

Conjecture Let $f = \sum h_{kl} y_k y_l \in \mathfrak{g}^2$. Denote $\Delta = \det(h_{kl})_{p+1 \leq k,l \leq n}$. Then

$$\nu = \dim\left(\frac{\mathcal{O}_\Sigma}{(\bar{b}\bar{\Delta})}\right).$$

If it is in this case then

$$\nu = \varrho(\Sigma X) + \delta_f.$$

By using a computer program *Singular* [GPS] we have checked that this conjecture is true for all the examples we know.

3.3 Blowing Up Some Line Singularities

Blowing up singularities is an important topic in geometry and a useful technique in singularity theory. Successively well chosen blowing ups give a resolution of singularities. The resolution data are important input for A'Campo's formulae on the zeta function and Lefschetz numbers (see [A'C]). Tibăr[Ti3] has observed that these formulae are also valid for functions on any analytic space. Gusein-Zade, Luengo and Melle in [GZLM] have developed an economical way to get the zeta function and Lefschetz number in some special cases by partial resolutions. Gonzalez-Sprinberg and Lejeune-Jalabert[GL1,2] have given a criterion on the existence of smooth curves on singular surfaces (see §3.3.1). A. Zaharia[Z3] has given a blowing-up characterization of simple isolated line singularities.

We first recall a result of Gonzalez-Sprinberg and Lejeune-Jalabert[GL1, 2]. Then we consider the blowing ups of some line singularities on surface with A_k singularity. We characterize the simple line singularities on A_1 surface by the behavior of their liftings under the resolution of the surface, which is a generalization of [Z3]. We give some examples to show how to get a normal crossing divisor by partial resolution, since this also helps us to get the zeta function and Euler characteristic, and to know the connectedness of the Milnor fibre.

3.3.1 Let X be a reduced pure dimensional surface embedded in some smooth space S, O a singular point of X. Let $p : \tilde{X} \longrightarrow X$ be a resolution of X. The *maximal cycle* $Z_{\tilde{X}} = \sum m_i e_i$ is a divisor in $\tilde{X}$ defined by the *partie divisorrielle* of $\mathfrak{m}_X \mathcal{O}_{\tilde{X}}$, where $\mathfrak{m}_X$ is the maximal ideal of $\mathcal{O}_{X,O}$, the e_i's are different irreducible components of the exceptional divisor $p^{-1}(O)$, and the m_i's are non-negative integers. If X is a rational singularity, the maximal cycle coincides with the *fundamental cycle*.

Theorem ([GL1] Proposition 1.1 and Proposition 3.1, see also [GL2]) Assume that $p^{-1}(O)$ has no isolated point. Then

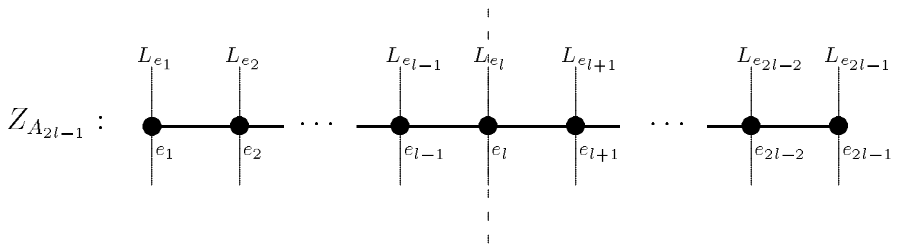
1) X has a smooth curve if and only if $Z_{\tilde{X}}$ has a reduced component, namely, $m_i = 1$ for some i;

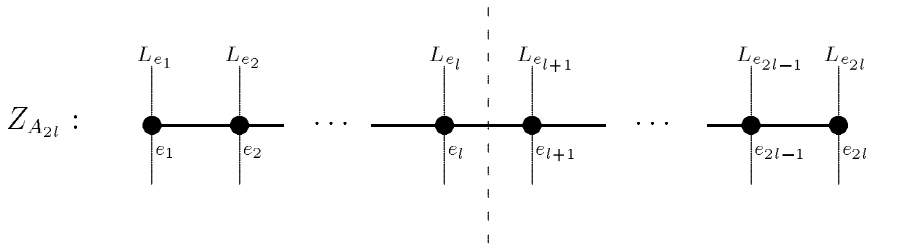
2) If Γ is a smooth curve on X, x is an exceptional point of the strict transform $\tilde{\Gamma}$ of Γ, then in a neighborhood of x, $\mathfrak{m}_X \mathcal{O}_{\tilde{X}}$ is locally principal. Any such a point is a smooth point of $p^{-1}(O)$ and belongs to some reduced irreducible component of $Z_{\tilde{X}}$. Conversely, for every point x on a reduced irreducible component of $Z_{\tilde{X}}$ in which $p^{-1}(O)$ is smooth, there exists a smooth curve Γ such that $x \in \tilde{\Gamma}$;

3) Consequently, to every reduced irreducible component e of $Z_{\tilde{X}}$ there corresponds a family L_e of smooth curves on X, and for different reduced irreducible components $e \neq e'$, $L_e \cap L_{e'} = \emptyset$.□

Remark As a consequence, there exist k, 3, 2 and 1 families of smooth curves on A_k $(k \geq 1)$, D_k $(k \geq 4)$, E_6 and E_7 surfaces respectively. And there are no smooth curves on the E_8 surface. In our classification of smooth curves on $A - D - E_6 - E_7$ surfaces in §1.4, there exist $\left[\frac{k+1}{2}\right]$, 2, 1 and 1 class(es) smooth curves on A_k $(k \geq 1)$, D_k $(k \geq 4)$, E_6 and E_7 surfaces respectively. The reason for this is the symmetry property of the simple singularities.

Let us consider A_k surfaces defined by $xy + z^{k+1} = 0$, $Z_{A_k} = e_1 + \cdots + e_k$. A calculation shows that the k families of smooth curves can be represented by $L_{e_i} : x = ut^i, y = -vt^{k-i+1}, z = t$ $(i = 1, \ldots, k)$, where $uv = 1$. We can arrange the indices of e_i's, so that any successive pair of e_i's has a normal crossing and draw the dual graph of Z_{A_k}, in which every vertex represents a component of Z_{A_k} and each edge indicates a normal crossing of the two components represented by its end points. The graph is symmetric with respect to the dashed line in the middle. The line passing through the vertex e_i is the set L_{e_i}. For every i, and every $\Sigma_i \in L_{e_i}$, we have $\varrho(\Sigma_i X) = \varrho(\Sigma_{k-i+1} X) = i$. Hence they belong to the same class in §1.4. This simply tells us that smooth curves on A_k surface with the same torsion number are symmetric.





In blowing-up an A_{2l-1} surface, we find the following phenomenon: after first blowing up, we can see that each line in $L_{e_1} \cup L_{e_{2l-1}}$ has a normal crossing with the exceptional divisors appeared. After i-th blowing up, we can see that each line in $L_{e_i} \cup L_{e_{2l-i}}$ has a normal crossing with the new exceptional divisors. And after the last blowing up, we see that each line in L_{e_l} and the last exceptional divisor e_l have normal crossing. A similar phenomenon happens to A_{2l} surfaces.

3.3.2 We recall the blowing up process. Let $(X, 0)$ be the germ of a surface in $(\mathbb{C}^3, 0)$ with isolated singularity at 0, $(\Sigma, 0) \subset (X, 0)$ a curve. Let $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ be a function with singular locus $\Sigma_f \supset \Sigma$, $H = f^{-1}(0)$. Let X be defined by $h = 0$. Let $p : M \longrightarrow X$ be the blowing up of X with center 0 and

$$M = \{(x, y, z, u_1 : u_2 : u_3) \mid xu_2 - yu_1 = xu_3 - zu_1 = yu_3 - zu_2 = h = 0\}.$$

$M \subset \mathbb{C}^3 \times \mathbb{P}^2$ is covered by three charts:
$M_1 = \{(x, y, z, u, v) \mid y = ux, z = vx, h = 0\} \quad (u_1 \neq 0)$;
$M_2 = \{(x, y, z, u', v') \mid x = u'y, z = v'y, h = 0\} \quad (u_2 \neq 0)$;
$M_3 = \{(x, y, z, s, t) \mid x = sz, y = tz, h = 0\} \quad (u_3 \neq 0)$.
Let $\tilde{X}, \tilde{H}$ and $\tilde{\Sigma}$ be the strict transform of X, H and Σ respectively:
$\tilde{X}$ = closure of $\{(x, y, z, u_1 : u_2 : u_3) \in M \mid (x, y, z) \neq 0\}$;
$\tilde{H}$ = closure of $\{(x, y, z, u_1 : u_2 : u_3) \in M \mid (x, y, z) \neq 0, f(x, y, z) = 0\}$;
$\tilde{\Sigma}$ = closure of $\{(x, y, z, u_1 : u_2 : u_3) \in M \mid 0 \neq (x, y, z) \in \Sigma\}$.
Let $\tilde{E} = \pi^{-1}(0)$, the exceptional divisor of M and $\tilde{E}' := \pi^{-1}(0) \cap \tilde{X}$, the exceptional divisor of $\tilde{X}$.

3.3.3 In this subsection, we blow up all the simple germs with line singularities on the surface X_1 with isolated A_1 singularity at 0 defined by $h = xy + z^2$. In this special case we have

$$\begin{aligned} M = \{&(x, y, z, u_1 : u_2 : u_3) \mid \\ &xu_2 - yu_1 = xu_3 - zu_1 = yu_3 - zu_2 = xy + z^2 = 0\}. \end{aligned}$$

$\tilde{X}_1$ is smooth and covered by three charts:

- $\tilde{X}_{11} = \{(x, y, z, u, v) \mid u + v^2 = 0, y = v^2 x, z = vx\}$ with x, v as local coordinates;
- $\tilde{X}_{12} = \{(x, y, z, u', v') \mid u' + v'^2 = 0, x = v'^2 y, z = v'y\}$ with y, v' as local coordinates;
- $\tilde{X}_{13} = \{(x, y, z, s, t) \mid st + 1 = 0, x = sz, y = tz\}$ with z, t as local coordinates.

Theorem Table 5 gives a summary of the descriptions of $\tilde{H}, \tilde{\Sigma}$ and the exceptional divisor $\tilde{E}_1'$ in $\tilde{X}_1$, where e is the multiplicity of $\tilde{E}_1'$, σ is the multiplicity of $\tilde{f}$ along $\tilde{\Sigma}$. The descriptions are in a neighborhood of $\tilde{E}_1'$. A_0 means that one component of $\tilde{H}$ is a smooth curve which intersects $\tilde{E}_1'$ transversally, and A_∞ means that a line, with multiplicity 2, intersects $\tilde{E}_1'$ transversally.

Table 5

Name	$\tilde{H}$ on $\tilde{X}$	σ	e	$\zeta_f(t)$	Λ_-
A_1A_∞	one A_∞	2	1	$(1-t)^{-1}$	1
A_1A_1	one A_∞, two A_0's	2	2	$1-t^2$	0
$A_1A_l(l>1)$	one A_∞ one A_{l-2}	2	2	$1-(-t)^{l+1}$	0
$A_1J_{l,\infty}$	one $J_{l,\infty}$, one A_0	2	2	$1-t^{3l+2}$	0
A_1D_∞	one D_∞,	2	2	$(1+t^2)(1+t^4)$	0

The proof of this theorem is an easy exercise of blowing up, we omit it. For the zeta function $\zeta_f(t)$ and Lefschetz number Λ_- see §3.3.6. $\square$

3.3.4 The properties of $\tilde{H}, \tilde{\Sigma}$ and $\tilde{E}_1'$ listed in the table above are sufficient to characterize the simple germs with line singularities on X_1.

Let $f : (X_1, 0) \longrightarrow (\mathbb{C}, 0)$ be a function with singular locus Σ defined by the ideal $\mathfrak{g} = (y, z)$. Then $\int \mathfrak{g} = (y, z^2)$. Let $f \in \int \mathfrak{g}$, then

$$f = a_0 y + ay^2 + 2byz + cz^2 \quad \text{with } a_0 \in \mathbb{C}, a, b, c \in \mathcal{O}$$

Let

$$A = \begin{pmatrix} a(0) & b(0) \\ b(0) & c(0) \end{pmatrix}.$$

Theorem Let $f \in \int \mathfrak{g} = (y, z^2)$.

1) If $e = 1$, then $f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y \quad (A_1A_\infty)$

2) If $e = 2$, rank(A)=2 and $\tilde{H}$ has one double lines (A_∞) and two lines (A_0's) intersecting with the exceptional divisor transversally, then

$$f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + z^2 \quad (A_1A_1);$$

3) If $e = 2$, rank(A)=2 and $\tilde{H}$ has $J_{l,\infty}$ type singlarity and a smooth line on $\tilde{X}_1$ intersecting with the exceptional divisor transversally, then

$$f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} yz + x^l z^2 \quad (A_1J_{l,\infty});$$

4) If $e = 2$, rank(A)=1 and $\tilde{H}$ has A_{l-2} type singularity on $\tilde{X}_1$, then

$$f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^{l+1} + z^2 \quad (A_1 A_l)(l > 1);$$

5) If $e = 2$, rank(A)=1 and $\tilde{H}$ has D_∞ type singularity on $\tilde{X}_1$, then

$$f \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y^2 + xz^2 \quad (A_1 D_\infty).$$

6) If f does not satisfy any of the above five conditions, then f is not simple.

Proof This is just re-do the classification of §1.5 using the conditions here. $\square$

3.3.5 If we blow up the singularities successively in a clever way, we can get an resolution:

$$p : (\tilde{X}, p^{-1}(H)) \longrightarrow (X, H)$$

such that $p^{-1}(H)$ is a normal crossing divisor. The strict transform $\tilde{H}$ of H can have several components with multiplicities: $\tilde{H} = \sigma\tilde{\Sigma} + \sum h_j \tilde{H}_j$. $S = p^{-1}(0) = \sum \lambda_j E_j$. So we have a decomposition of $p^{-1}(H)$ into irreducible components with multiplicities:

$$p^{-1}(H) = \sigma\tilde{\Sigma} + \sum h_j \tilde{H}_j + \sum \lambda_j E_j$$

in $\tilde{X}$.

Let (see [A'C] and [Ti3])

$$S_i = \left\{ s \in S \mid \text{ local equation of } p^{-1}(H) \text{ in } s \in \tilde{X} \text{ is } z^i = 0 \right\}.$$

Then the zeta function $\zeta_f(t)$ of the monodromy of the Milnor fibre F_f of f, the Lefschetz number $\Lambda(h_f^k)$ of the monodromy transformation and the Euler characteristic $\chi(F_f)$ are given by the following formulae which have been proved by A'Campo [A'C] for smooth X and generalized to the general case by Tibăr [Ti3]:

$$\zeta_f(t) = \prod_{i \geq 1} (1 - t^i)^{-\chi(S_i)}, \ \Lambda(h_f^k) = \sum_{i|k} i \cdot \chi(S_i) \ (k \geq 1), \ \chi(F_f) = \sum_{i \geq 1} i \cdot \chi(S_i).$$

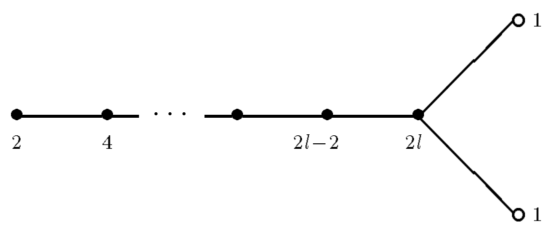
3.3.6 Partial resolution and normal crossing divisor In order to produce the p in §3.3.5, one should first resolve X such that the exceptional divisor is a normal crossing divisor. Then one should resolve the lifting of the divisor $f = 0$ into a normal crossing divisor. In some special cases, one will not need the second resolutions (or part of them) if one can derive the zeta function $\zeta_f(t)$ from the lifting of f. This is called partial resolution (see [GZLM]).

In this subsection, we give the resolutions of some line singularities on an A_k surface. We denote by $\bullet$ the exceptional divisor of p, by $\circ$ the divisor H_i, and by double circle the divisor $\tilde{\Sigma}$. The number attached to each divisor is the multiplicity of $f \circ p$ along the divisor. If two divisors intersect each other, we draw a line connecting them. Then we get a graph G_f of $p^{-1}(H)$, called the *total resolution graph* of $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$. This kind of graph is not standard but has some advantages. For example, we can see clearly the normal crossing divisors of the blowing up and get easily the Euler characteristic and zeta function. Also in some cases, G_f can be obtained by partial resolution. We give some examples to explain this.

Example 1 Let $X = \mathbb{C}^2$.

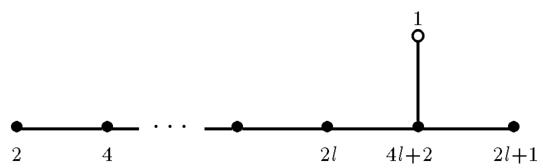
1) For plane curve with A_k type singularity defined by $f_k = x^{k+1} + y^2$, successive blowing ups give the following graphs. In doing this, one finds that the strict transform of $A_k (k > 2)$ after one blowing up is of type A_{k-2}. So one can get G_{A_k} from $G_{A_{k-2}}$ by just puting an extra exceptional divisor with multiplicity 2 to the left hand side of $G_{A_{k-2}}$, and adding 2 or 4 to the multiplicities of $G_{A_{k-2}}$.

$G_{A_{2l-1}}$:



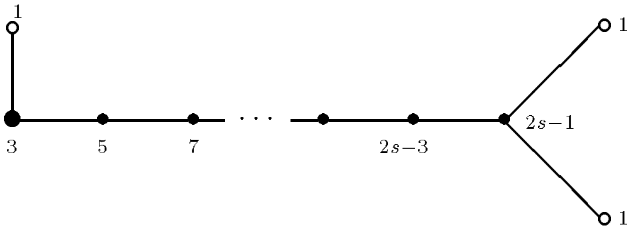
$\zeta_{A_{2l-1}} = \frac{1-t^{2l}}{1-t^2}$

$G_{A_{2l}}$:



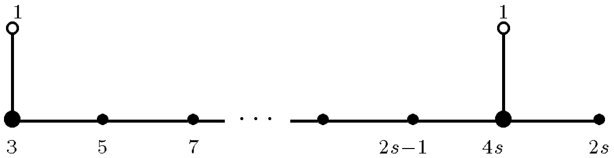
$\zeta_{A_{2l}} = \frac{1+t^{2l+1}}{1-t^2}$

$G_{D_{2s}}$:



$\zeta_{D_{2s}} = 1 - t^{2s-1}$

$G_{D_{2s+1}}$:



$\zeta_{D_{2s+1}} = 1 + t^{2s}$

2) After one blowing up of $D_k (k \geq 6)$ type singularity $f_k = y^{k-1} + x^2 y$, one finds that the strict transform of $f^{-1}(0)$ consists of an A_{k-5} type singularity and a smooth line intersecting the exceptional divisor transversally, and the exceptional divisor has multiplicity 3. One can get the total resolution graph G_{D_k} from that of A_{k-5} by adding 3 (or 6) to the multiplicities of the exceptional divisor and additional exceptional divisor (denoted by a bigger bullet in the graph) with multiplicity 3 at the left hand side.

Example 2 Let X_k be an A_k surface.

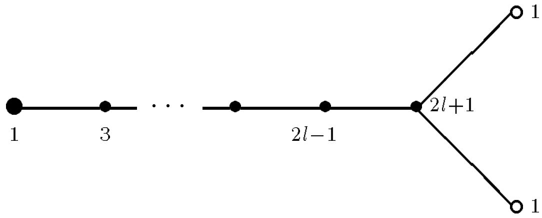
We recall a list from [Di] or [Ly]. All the $\mathcal{R}$-simple functions with isolated singularity on A_k surfaces are contained in the table 6.

Table 6

Name	Type of X	Normal form	μ	$\zeta_f(t)$	Λ
C_{p+1}	$A_1 :\ xy + z^2$	$x + y^p\ (p \geq 1)$	p	$(1-t)^{-1}(1-(-t)^{p+1})$	1
B_3	$A_2 :\ xy + z^3$	z	1	1	0
F_4	$A_2 :\ xy + z^3$	$x + y$	2	$(1-t)^{-2}(1-t^3)$	2
B_{k+1}	$A_k :\ xy + z^{k+1}$	$z\ (k \geq 3)$	1	0	1

The the followings are the total resolution graphs of the normal forms in table 6. We use Λ_{-} to denote the first Lefschtez number, and ζ_{-} to denote the zeta function:

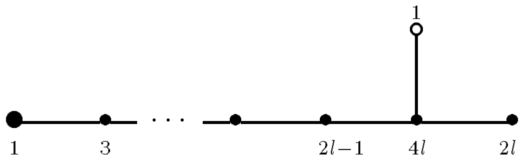
$G_{C_{2l}}$:



$\Lambda_{C_{2l}} = 1$

$\zeta_{C_{2l}} = \frac{1-t^{2l+1}}{1-t}$

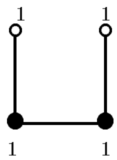
$G_{C_{2l+1}}$:



$\Lambda_{C_{2l-1}} = 1$

$\zeta_{C_{2l-1}} = \frac{1+t^{2l}}{1-t}$

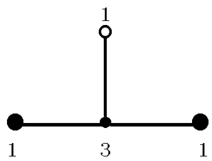
G_{B_3} :



$\Lambda_{B_3} = 0$

$\zeta_{B_3} = 1$

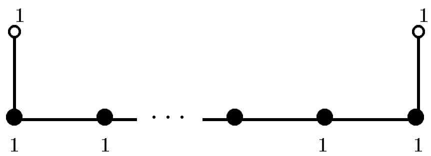
G_{F_4} :



$\Lambda_{F_4} = 2$

$\zeta_{F_4} = \frac{1-t^3}{(1-t)^2}$

$G_{B_k}(k \geq 4)$:



$\Lambda_{B_k} = 0$

$\zeta_{B_k} = 1$

Example 3 Let X_k be an A_k surface, and $f : X_k \longrightarrow \mathbb{C}$ an analytic function with singular locus Σ a line in X_k. Assume that Σ and X have minimal torsion number, and f is a simple function in Table 2 and 3. We give the total resolution graph of f.

First we consider the case $k = 1$.

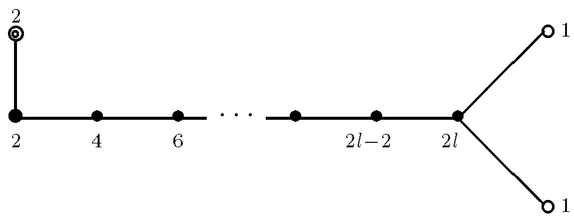
Since the strict transform of A_1A_l under the resolution of X_1 consists of a double line ($\tilde{\Sigma}$) and an A_{l-2}, we have (compare $G_{A_1A_l}$ with $G_{A_{l-2}}$, the bigger bullet denoted the exceptional divisor coming from the resolution of X_1):

$G_{A_1A_\infty}$:

$$\Lambda_{A_1A_\infty} = 1$$

$$\zeta_{A_1A_\infty} = \frac{1}{1-t}$$

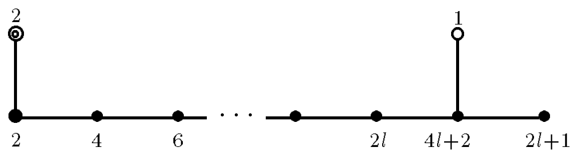
$G_{A_1A_{2l-1}}$:



$$\Lambda_{A_1A_{2l-1}} = 0$$

$$\zeta_{A_1A_{2l-1}} = 1 - t^{2l}$$

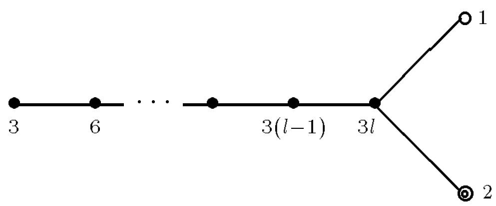
$G_{A_1A_{2l}}$:



$$\Lambda_{A_1A_{2l}} = 0$$

$$\zeta_{A_1A_{2l}} = 1 + t^{2l+1}$$

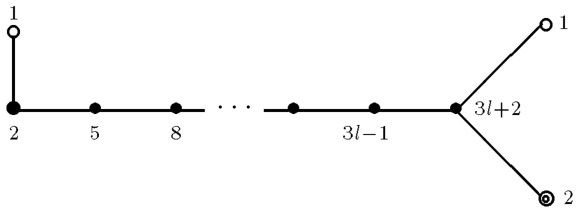
$G_{J_{l,\infty}}$:



$$\Lambda_{J_{l,\infty}} = 0$$

$$\zeta_{J_{l,\infty}} = \frac{1-t^{3l}}{1-t^3}$$

$G_{A_1 J_{l,\infty}}$:

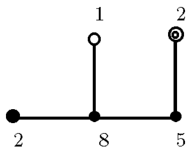


$\Lambda_{A_1 J_{l,\infty}} = 0$

$\zeta_{A_1 J_{l,\infty}} = 1 - t^{3l+2}$

For $A_1 J_{l,\infty}$, the strict transform of it under the resolution of X_1 has a smooth line intersecting the exceptional divisor transversally and a singularity $J_{l,\infty} : y^2(x^k + y)$. If we blow up $J_{l,\infty}$ with $(0,0)$ as center, we have $G_{J_{l,\infty}}$. From this we know $G_{A_1 J_{l,\infty}}$.

$G_{A_1 D_\infty}$:

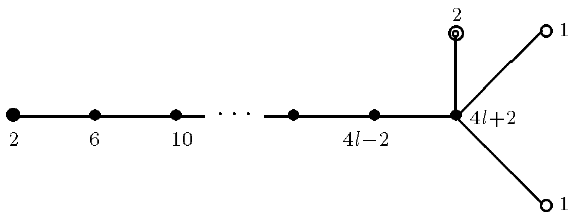


$\Lambda_{A_1 D_\infty} = 0$

$\zeta_{A_1 D_\infty} = (1 + t^2)(1 + t^4)$

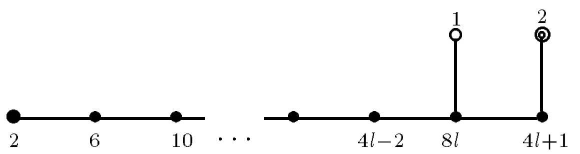
For $f_l = y^2 + x^l z^2$ on X_1 (note that this germs is not simple), the lifting of f by the resolution of X_1 is $x^2 v^2 (v^2 + x^l)$. From $G_{A_{l-1}}$ we get:

$G_{f_{2l}}$:



$\zeta_{f_{2l}} = \frac{(1-t^{4l+2})^2}{1-t^2}$

$G_{f_{2l-1}}$:

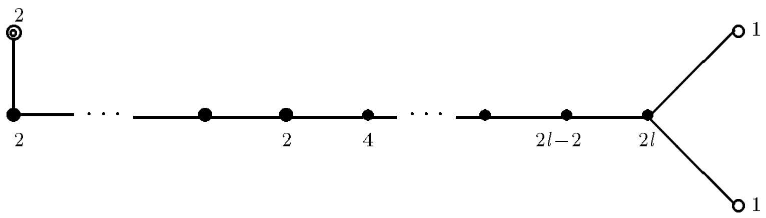


$\zeta_{f_{2l-1}} = \frac{1-t^{8l}}{1-t^2}$

Next, we consider $k \geq 2$ cases.

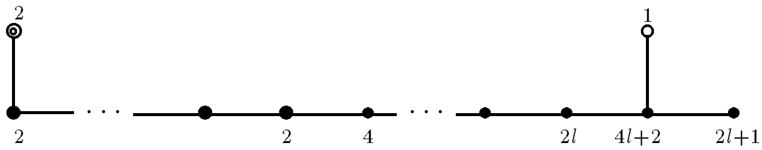
In the same way, for $A_k A_l : f_l = y^{l+1} + z^2$ on $X_k : xy + z^{k+1}$, we get their graphs, where the bigger bullets denote the exceptional divisor "coming from " the resolution of X_k, the number of them is k. Note that the Euler characteristic of the Milnor fibre and the zeta function of an $A_k A_l$-type germ do not depend on k, but the total resolution graph depend on k.

$G_{A_kA_{2l-1}}$:



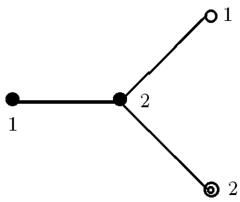
$$\Lambda_{A_kA_{2l-1}} = 0, \quad \zeta_{A_kA_{2l-1}} = 1 - t^{2l}$$

$G_{A_kA_{2l}}$:

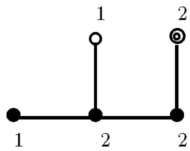


$$\Lambda_{A_kA_{2l}} = 0, \qquad \zeta_{A_kA_{2l}} = 1 + t^{2l+1}$$

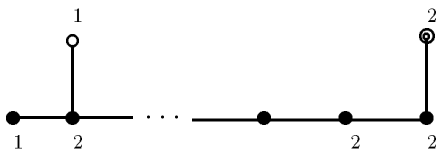
$G_{A_2A_\infty}$:



$G_{A_3A_\infty}$:



$G_{A_kA_\infty}$:
$(k \geq 4)$

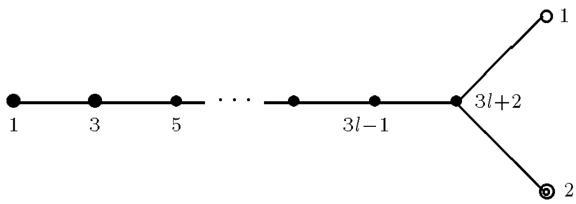


$$\Lambda_{A_kA_\infty} = 1$$

$$\zeta_{A_kA_\infty} = 1 + t$$

$(k \geq 2)$

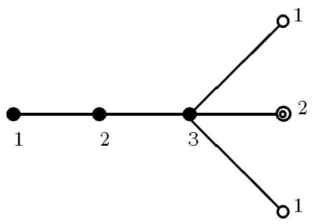
$G_{A_2J_{l,\infty}}$:



$$\Lambda_{A_2J_{l,\infty}} = 1$$

$$\zeta_{A_2J_{l,\infty}} = \frac{1-t^{3l+2}}{1-t}$$

$G_{A_3D_\infty}$:



$$\Lambda_{A_3D_\infty} = 1$$

$$\zeta_{A_3D_\infty} = \frac{(1-t^3)^2}{1-t}$$

3.3.7 Remark In the total resolution graphs in §3.3.6, we find that each graph has at least one *reduced component*, the component with multiplicity 1. This fact tells us that the Milnor fibre F of f is 0-connected. In general, if the multiplicities in the total resolution graph do not have common divisor except 1, then F is 0-connected. This is useful when $\dim X = 2$, since this tells us that F is a bouquet of circles. Hence the only question is to know the number of the circles, the Milnor number of f, but this follows from the Euler characteristic. In order to know if the maximal common divisor of the multiplicities is 1, one can also use partial resolution. Especially, if one can find a reduced component of $f^{-1}(0)$, then F is 0-connected. This can be seen even without blowing up.

Example If X is the E_6 surface defined by $x^2y-y^2+z^3=0$, at the origin, $f_s \overset{\mathcal{R}_{\Sigma X}}{\leftrightsquigarrow} y+tz^2 \quad (t \neq 0)$ (*loc. cit.*). From the equations, we find that H can be defined by $y+tz^2=0$ and $z^2(z-tx^2-t^2z^2)=0$, there is a reduced component in H. Hence F^c is a bouquet of circles. By the formula in §3.2, we know that $\chi(F) = -3$,

$$\begin{aligned} F^c =& \{(x,y,z) \in \mathbb{C}^3 \mid x^2y-y^2+z^3=0, y+tz^2=\eta, \\ & |x|^2+|y|^2+|z|^2 \leq \epsilon\} \overset{h}{\simeq} S^1 \vee S^1 \vee S^1 \vee S^1, \quad \text{and} \\ F \overset{h}{\simeq}& S^1 \vee \cdots \vee S^1 \quad (\mu = {}^\sharp A_1 + 2{}^\sharp D_\infty + 4 \text{ copies}). \end{aligned}$$

If we use the similar method to the germs in table 4 in §1.5, we find the Milnor fibre of each of the germs is a bouquet of circles.

Theorem Let X be a surface with isolated singularity of type A_k $(k \geq 1)$, D_k $(k \geq 4)$, E_6, or E_7. Let f be a function on X defining a line singulariy with line Σ as its singular locus (no conditions on torsion number). If the transversal singularity of f along Σ is of type A_1, then the Milnor fibre F of f is homotopy equivalent to a bouquet of circles. And the number of the circles is

$$\mu = \mu^c +^\sharp A_1 + 2^\sharp D_\infty.$$

where the central Milnor number μ^c is the Milnor number of the stable germs (if there exist) or that of the germs with minimal codimensions.$\square$

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Samenvatting in het Nederlands

Dit proefschrift is een studie van functies op complexe singuliere ruimten met waarden in $\mathbb{C}$. In het bijzonder behandelt het functies met niet-geïsoleerde singulariteiten. Het bouwt enerzijds voort op het werk van Dimca, Bruce-Roberts, Wall en Tibăr over functies met geïsoleerde singulariteiten op singuliere ruimten en anderzijds op het werk van Siersma, Pellikaan, Van Straten, de Jong en Zaharia over functies met niet-geïsoleerde singulariteiten op gladde ruimten. Centrale onderwerpen in het proefschrift zijn:

- classificatie,
- deformatie en Morsificatie,
- topologie van de Milnorvezel.

De methoden zijn afkomstig uit de algebra, analyse, meetkunde en topologie.

Het proefschrift is een eerste verkenning op het gebied. Daarom worden veelal speciale gevallen onderzocht. Meestal heeft de complexe ruimte X zelf een geïsoleerde singulariteit en de functie $f : X \to \mathbb{C}$ een 1-dimensionale singuliere locus Σ. Soms is Σ een gladde complexe lijn.

Hoofdstuk 1 behandelt oa. de classificatie van functies met eindige codimensie en de equivalente karakterisering: transversale A_1-singulariteit. Daarnaast wordt ingegaan op de verschillende posities van lijnen op oppervlakken van type A-D-E.

Hoofdstuk 2 behandelt de deformatietheorie. Hierin speelt de relatieve positie van Σ in X een belangrijke rol. Dit vertaalt zich in vragen rond het conormale moduul, betreffende torsie, en het bestaan van reguliere rijtjes, die Σ definieren en zich goed gedragen ta.v. de definiërende vergelijkingen van X. Voor diverse gevallen wordt de existentie van een 'Morsificatie' bewezen, oa. voor het lastiger geval Σ een gladde lijn.

Hoofdstuk 3 behandelt de topologie van de Milnorvezel. Met behulp van de Morsificatie worden allereerst in de homologie bouquetstellingen bewezen met in speciale gevallen een homotopieversie. Daarnaast zijn er formules voor de Euler-karakeristiek en tenslotte een beschrijving van de monodromie met behulp van de A'Campo-methode.

Summary in Chinese

Curriculum Vitae

The author was born at Dalin, Liaoning of China, on 23 July, 1959. He went to schools during the " cultural revolution", and became a farmer in August 1977.

At the end of 1977, he passed the first entrance examination to the universities in China after the "revolution". Between March, 1978 and January 1982, he studied mathematics at the department of mathematics of Jinzhou Normal University. After his graduation, he became a teaching assistant at the same department.

He began his graduate studies in September 1983 at the department of mathematics of Northeast Normal University. He learnt topology and singularity theory and its applications (catastrophe theory). He got master degree in July 1986 when he finished his graduate study.

Since August 1986, he has been a teacher in Jinzhou Normal University. Between Feburary 1992 and August 1993, he made his further studies at the department of mathematics of Utrecht University, the Netherlands. He joint the first master class of Mathematical Research Institute to study geometry, and was awarded the master class certificate.

From September 1993, he started to prepare his Ph.D. thesis on singularity theory with Prof. dr. Dirk Siersma as supervisor. The preparation took place partly in Jinzhou and in Utrecht.

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